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Stability of higher-order nonlinear impulsive differential equations

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Abstract

For a higher-order nonlinear impulsive ordinary differential equation, we present the concepts of Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and generalized Hyers–Ulam–Rassias stability. Furthermore, we prove the generalized Hyers–Ulam–Rassias stability by using integral inequality of Grönwall type for piecewise continuous functions. These results extend related contributions to the corresponding first-order impulsive ordinary differential equation. Hyers–Ulam stability, generalized Hyers–Ulam stability, and Hyers–Ulam–Rassias stability can be discussed by the same methods.

Keywords: Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias stability, generalized Hyers–Ulam–Rassias stability, nonlinear impulsive differential equation, higher-order, Grönwall inequality.
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1. Introduction

The stability theory is an important branch of the qualitative analysis of differential equations. In particular, for the stability of functional equations, Ulam [25] raised a question: "When can an approximate

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homomorphism from a group G_1 to a metric group G_2 be approximated by an exact homomorphism?"

To solve this question, assuming that G_1 and G_2 are Banach spaces and using a direct method, Hyers [7] brilliantly gave a partial answer. This result was then extended and improved by Aoki [2] and Rassias [22] who weakened the condition for the bound of the norm of Cauchy difference. For further details and discussion, see the monograph by Jung [11].

As far as we know, works by Obłoza [17, 18] were among the first contributions dealing with the Hyers–Ulam stability of differential equations. Since then, Hyers–Ulam stability and Hyers–Ulam–Rassias stability of various classes of differential equations and differential operators have been explored by using a wide spectrum of approaches; see, e.g., [1, 4–6, 8–10, 12, 13, 16, 20, 21, 24, 34] and the references cited therein.

In recent years, Hyers–Ulam stability and Hyers–Ulam–Rassias stability of impulsive differential equations have always attracted interest of researchers; see, for instance, [3, 14, 19, 26, 27, 33]. One of the main reasons for this lies in the fact that, as pointed out by Lupulescu and Zada [15], Rogovchenko [23], Wang and Liu [28], Wang and Wu [29], and Wang et al. [30–32], impulsive differential equations arise in a number of applied problems in natural sciences and engineering. Note that the results reported in [3, 14, 19, 26, 27, 33] are concerned with several classes of first-order impulsive differential equations. Thereinto, Wang et al. [26] introduced four Ulam's type stability (Hyers–Ulam stability, generalized Hyers–Ulam stability, Hyers–Ulam–Rassias stability, and generalized Hyers–Ulam–Rassias stability) concepts for a first-order impulsive ordinary differential equation. So far, to the best of our knowledge, Ulam's type stability results of higher-order impulsive ordinary differential equations have not been studied yet.

It should be noted that research in this paper was strongly motivated by the recent contributions of Wang et al. [26]. Our principal goal is to analyze the Ulam's type stability of the higher-order impulsive differential equation

$$\begin{cases} y^{(n)}(t) = F(t, y(t), y'(t), y''(t), \dots, y^{(n-1)}(t)), & t \in I' = I\backslash\{t_1, t_2, \dots, t_m\}, \\ \Delta y^{(i)}(t_k) = y^{(i)}(t_k^+) - y^{(i)}(t_k^-) = \Upsilon_k(y^{(i)}(t_k^-)), & i = 0, 1, \dots, n-1 \text{ and } k = 1, 2, \dots, m, \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ y''(t_0) = y_2,\ \dots,\ y^{(n-1)}(t_0) = y_{n-1}, \end{cases} \tag{1.1}$$

where $n \geq 1$ is a natural number, $I = [t_0, t_F]$, t_k satisfy $0 \leq t_0 < t_1 < t_3 < \cdots < t_m < t_{m+1} = t_F < +\infty$, $F : B \to R$ is a continuous function on a closed ball B in $I \times R^n$, $\Upsilon_k : R \to R$ is a continuous function for each k, $y^{(i)}(t_k^+) = \lim_{\tau\to 0^+} y^{(i)}(t_k + \tau)$ and $y^{(i)}(t_k^-) = \lim_{\tau\to 0^+} y^{(i)}(t_k - \tau)$ represent the right-sided and left-sided limits of $y^{(i)}(t)$ at t_k, respectively.

2. Preliminaries

In this section, we present some definitions of Ulam's type stability and auxiliary lemmas to prove our main results. Throughout this paper, we use the following spaces:

- $C(I, R)$ is the Banach space of all continuous functions from I to R with norm $\|x\|_C = \sup\{|x(t)| : t \in I\}$;
- $PC(I, R)$ denotes the Banach space of all functions $x : I \to R$ with norm $\|x\|_{PC} = \sup\{|x(t)| : t \in I\}$ such that $x \in C((t_k, t_{k+1}], R)$, $k = 0, 1, \dots, m$ and there exist $x(t_k^+)$ and $x(t_k^-)$ satisfying $x(t_k^-) = x(t_k)$, $k = 1, 2, \dots, m$;
- $PC^n(I, R) = \{x : I \to R \,|\, x^{(i)} \in PC(I, R), i = 0, 1, \dots, n\}$ is the Banach space with norm $\|x\|_{PC^n} = \max\{\|x^{(i)}\|_{PC} : i = 0, 1, \dots, n\}$.

Let $R^+ = [0, +\infty)$, $\{y\} = (y, y', y'', \dots, y^{(n-1)})$, $\epsilon > 0$, $\mu \geq 0$, and $\theta \in PC(I, R^+)$ be nondecreasing. We focus on the following inequalities:

$$\begin{cases} |y^{(n)}(t) - F(t, \{y\})| \leq \epsilon, & t \in I', \\ |\Delta y^{(i)}(t_k) - \Upsilon_k(y^{(i)}(t_k^-))| \leq \epsilon, & i = 0, 1, \dots, n-1 \text{ and } k = 1, 2, \dots, m, \end{cases} \tag{2.1}$$

$$\begin{cases} |y^{(n)}(t) - F(t, \{y\})| \leq \theta(t), & t \in I', \\ |\Delta y^{(i)}(t_k) - \Upsilon_k(y^{(i)}(t_k^-))| \leq \mu, & i = 0, 1, \ldots, n-1 \text{ and } k = 1, 2, \ldots, m, \end{cases} \tag{2.2}$$

and

$$\begin{cases} |y^{(n)}(t) - F(t, \{y\})| \leq \epsilon\theta(t), & t \in I', \\ |\Delta y^{(i)}(t_k) - \Upsilon_k(y^{(i)}(t_k^-))| \leq \epsilon\mu, & i = 0, 1, \ldots, n-1 \text{ and } k = 1, 2, \ldots, m. \end{cases} \tag{2.3}$$

In what follows, we introduce the concepts of Ulam's type stability of (1.1).

Definition 2.1. Equation (1.1) is said to be Hyers–Ulam stable on I if there exists a real number $K_{F,m} > 0$ such that, for every $\epsilon > 0$ and for every solution $y \in PC^n(I, R)$ of (2.1), there exists a solution $x_0 \in PC^n(I, R)$ of (1.1) with

$$|y(t) - x_0(t)| < K_{F,m}\epsilon, \qquad \text{for } t \in I.$$

Definition 2.2. Equation (1.1) is called generalized Hyers–Ulam stable on I if there is a function $G_{F,m} \in C(R^+, R^+)$ with $G_{F,m}(0) = 0$ such that, for every $\epsilon > 0$ and for every solution $y \in PC^n(I, R)$ of (2.1), there exists a solution $x_0 \in PC^n(I, R)$ of (1.1) with

$$|y(t) - x_0(t)| < G_{F,m}(\epsilon), \qquad \text{for } t \in I.$$

Definition 2.3. Equation (1.1) is termed Hyers–Ulam–Rassias stable on I with respect to (θ, μ) if there exists an $M_{F,m,\theta} > 0$ such that, for every $\epsilon > 0$ and for every solution $y \in PC^n(I, R)$ of (2.3), there exists a solution $x_0 \in PC^n(I, R)$ of (1.1) with

$$|y(t) - x_0(t)| < M_{F,m,\theta}\epsilon(\theta(t) + \mu), \qquad \text{for } t \in I.$$

Definition 2.4. Equation (1.1) is said to be generalized Hyers–Ulam–Rassias stable on I with respect to (θ, μ) if there exists an $L_{F,m,\theta} > 0$ such that, for every solution $y \in PC^n(I, R)$ of (2.2), there exists a solution $x_0 \in PC^n(I, R)$ of (1.1) with

$$|y(t) - x_0(t)| < L_{F,m,\theta}(\theta(t) + \mu), \qquad \text{for } t \in I.$$

Remark 2.5. Definition 2.1 $\Rightarrow$ Definition 2.2; Definition 2.3 $\Rightarrow$ Definition 2.4; for $\theta(t) = \mu = 1$, Definition 2.3 $\Rightarrow$ Definition 2.1.

The following inequality is the well-known integral inequality of Grönwall type for piecewise continuous functions.

Lemma 2.6. *If*

$$x(t) \leq a(t) + \int_{t_0}^{t} b(s)x(s)ds + \sum_{t_0 < t_k < t} \xi_k x(t_k^-)$$

for $t \geq t_0 \geq 0$, where $x, a, b \in PC([t_0, \infty), R^+)$, a is nondecreasing, $b(t) > 0$, and $\xi_k > 0$, then

$$x(t) \leq a(t) \prod_{t_0 < t_k < t} (1 + \xi_k) \exp\left(\int_{t_0}^{t} b(s)ds\right)$$

for $t \geq t_0$.

Remark 2.7. It follows directly from inequality (2.1) that a function $y \in PC^n(I, R)$ satisfies (2.1) if and only if there is a function $f \in PC(I, R)$ and a sequence f_k^i (which defend on y) such that $|f(t)| \leq \epsilon$ for $t \in I$, $|f_k^i| \leq \epsilon$ for $i = 0, 1, \ldots, n-1$ and $k = 1, 2, \ldots, m$, and

$$\begin{cases} y^{(n)}(t) = F(t, \{y\}) + f(t), & t \in I', \\ \Delta y^{(i)}(t_k) = \Upsilon_k(y^{(i)}(t_k^-)) + f_k^i, & i = 0, 1, \ldots, n-1 \text{ and } k = 1, 2, \ldots, m. \end{cases}$$

Remark 2.8. If $y \in PC^n(I,R)$ satisfies (2.1), then

$$\left| y^{(n-i)}(t) - \sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} - \sum_{j=1}^{k} \Upsilon_j(y^{(n-i)}(t_j^-)) - \int_{t_0}^{t} \frac{(t-s)^{i-1}}{(i-1)!} F(s,\{y\})ds \right| \leq \left(\frac{(t-t_0)^i}{i!} + m \right)\epsilon,$$

where $t \in I$ and $i = 1, 2, \ldots, n$.

Proof. It follows from Remark 2.7 that, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} y^{(n-i)}(t) = {} & \sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} + \sum_{j=1}^{k} \Upsilon_j(y^{(n-i)}(t_j^-)) + \sum_{j=1}^{k} f_j^{n-i} + \int_{t_0}^{t} \frac{(t-s)^{i-1}}{(i-1)!} F(s,\{y\})ds \\ & + \int_{t_0}^{v_i=t} \ldots \int_{t_0}^{v_2} \int_{t_0}^{v_1} f(s)ds. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} & \left| y^{(n-i)}(t) - \sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} - \sum_{j=1}^{k} \Upsilon_j(y^{(n-i)}(t_j^-)) - \int_{t_0}^{t} \frac{(t-s)^{n-i}}{(i-1)!} F(s,\{y\})ds \right| \\ & \leq \int_{t_0}^{v_i=t} \ldots \int_{t_0}^{v_2} \int_{t_0}^{v_1} |f(s)|ds + \sum_{j=1}^{k} |f_j^{n-i}|, \end{aligned}$$

which implies that

$$\left| y^{(n-i)}(t) - \sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} - \sum_{j=1}^{k} \Upsilon_j(y^{(n-i)}(t_j^-)) - \int_{t_0}^{t} \frac{(t-s)^{i-1}}{(i-1)!} F(s,\{y\})ds \right| \leq \left(\frac{(t-t_0)^i}{i!} + m \right)\epsilon.$$

The proof is complete. □

Remark 2.9. One can obtain similar remarks for solutions of inequalities (2.2) and (2.3). The details are left to the reader.

3. Main results

Define a closed ball $B = I \times \prod_{i=0}^{n-1}[-M_i, M_i]$, where $M_i = \|y^{(i)}\|_{PC}$. In this section, we prove the Ulam's type stability of (1.1) with the condition

$$(t-t_0)^{n-1}\big|F(t,\{y\}) - F(t,\{z\})\big| \leq h(t)\big|y(t) - z(t)\big|, \tag{3.1}$$

where $h : I \to R^+$ is an integrable function and the Lipschitz condition

$$\big|F(t,\{y\}) - F(t,\{z\})\big| \leq S_0 \sum_{i=0}^{n-1} \big|y^{(i)}(t) - z^{(i)}(t)\big|, \qquad S_0 > 0 \text{ is a constant}, \tag{3.2}$$

respectively.

Theorem 3.1. *If*

(H$_1$) *F satisfies condition* (3.1)*;*

(H$_2$) *$\Upsilon_k : R \to R$ and there exist constants $M_k > 0$ such that $|\Upsilon_k(x_1) - \Upsilon_k(x_2)| \leq M_k|x_1 - x_2|$ for $k = 1, 2, \ldots, m$ and $x_1, x_2 \in R$;*

(H_3) *there exists a nondecreasing function $\theta \in PC(I, R^+)$ such that, for $t \in I$ and for some $\rho_\theta > 0$,*

$$\int_{t_0}^{t} \theta(s)ds \leq \rho_\theta \theta(t),$$

then (1.1) *has generalized Hyers–Ulam–Rassias stability on I with respect to (θ, μ). If, in addition,*

$$\frac{1}{(n-1)!}\int_{t_0}^{t_F} h(s)ds + \sum_{j=1}^{m} M_j < 1,$$

then (1.1) *has a unique solution in $PC^n(I, R) \cap PC(I, R)$.*

Proof. Let $y \in PC^n(I, R)$ be a solution to (2.2). The exact solution $x \in PC^n(I, R)$ of the initial value problem

$$\begin{cases} x^{(n)}(t) = F(t, \{x\}), \qquad t \in I', \\ \Delta x^{(i)}(t_k) = \Upsilon_k(x^{(i)}(t_k^-)), \qquad i = 0, 1, \dots, n-1 \text{ and } k = 1, 2, \dots, m, \\ x(t_0) = y_0,\ x'(t_0) = y_1,\ x''(t_0) = y_2, \dots, x^{(n-1)}(t_0) = y_{n-1}, \end{cases}$$

is given by

$$x(t) = \begin{cases} \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}F(s,\{x\})ds, \qquad t \in [t_0, t_1], \\ \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \Upsilon_1(x(t_1^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}F(s,\{x\})ds, \qquad t \in (t_1, t_2], \\ \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \sum_{j=1}^{2}\Upsilon_j(x(t_j^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}F(s,\{x\})ds, \qquad t \in (t_2, t_3], \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \sum_{j=1}^{m}\Upsilon_j(x(t_j^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}F(s,\{x\})ds, \qquad t \in (t_m, t_F]. \end{cases}$$

Similar as in Remark 2.8, an application of inequality (2.2) implies that, for $t \in I$,

$$\left| y(t) - \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} - \sum_{j=1}^{k}\Upsilon_j(y(t_j^-)) - \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}F(s,\{y\})ds \right| \leq (m + \rho_\theta^n)(\theta(t) + \mu).$$

Hence, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} \big|y(t) - x(t)\big| &\leq \left| y(t) - \sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} - \sum_{j=1}^{k}\Upsilon_j(y(t_j^-)) - \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}F(s,\{y\})ds \right| \\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}\big|F(s,\{y\}) - F(s,\{x\})\big|ds + \sum_{j=1}^{k}\big|\Upsilon_j(y(t_j^-)) - \Upsilon_j(x(t_j^-))\big| \\ &\leq (m + \rho_\theta^n)(\theta(t) + \mu) + \frac{1}{(n-1)!}\int_{t_0}^{t} h(s)\big|y(s) - x(s)\big|ds + \sum_{j=1}^{k} M_j\big|y(t_j^-) - x(t_j^-)\big|. \end{aligned}$$

By virtue of Lemma 2.6, we conclude that, for $t \in I$,

$$\big|y(t) - x(t)\big| \le (m + \rho_\theta^n)(\theta(t) + \mu) \prod_{t_0 < t_k < t} (1 + M_k) \exp\left(\frac{1}{(n-1)!}\int_{t_0}^{t} h(s)ds\right) \le L_{F,m,\theta}(\theta(t) + \mu),$$

where

$$L_{F,m,\theta} = (m + \rho_\theta^n) \prod_{k=1}^{m} (1 + M_k) \exp\left(\frac{1}{(n-1)!}\int_{t_0}^{t_F} h(s)ds\right).$$

Therefore, (1.1) is generalized Hyers–Ulam–Rassias stable on I with respect to (θ, μ).

Uniqueness of solution. For $g \in PC(I, R)$, define an operator $\Lambda : PC(I, R) \to PC(I, R)$ by

$$(\Lambda g)(t) = \begin{cases} \displaystyle\sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} F(s, \{g\})ds, & t \in [t_0, t_1], \\ \displaystyle\sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \Upsilon_1(g(t_1^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} F(s, \{g\})ds, & t \in (t_1, t_2], \\ \displaystyle\sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \sum_{j=1}^{2} \Upsilon_j(g(t_j^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} F(s, \{g\})ds, & t \in (t_2, t_3], \\ \cdot \\ \cdot \\ \cdot \\ \displaystyle\sum_{j=0}^{n-1} \frac{(t-t_0)^j y_j}{j!} + \sum_{j=1}^{m} \Upsilon_j(g(t_j^-)) + \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} F(s, \{g\})ds, & t \in (t_m, t_F]. \end{cases}$$

Clearly, Λ is well-defined. We show that Λ is a Picard operator on $PC(I, R)$. For this, let $g_1, g_2 \in PC(I, R)$ and consider

$$\begin{aligned} \big|(\Lambda g_1)(t) - (\Lambda g_2)(t)\big| &= \Bigg| \sum_{j=1}^{k} \big(\Upsilon_j(g_1(t_j^-)) - \Upsilon_j(g_2(t_j^-))\big) \\ &\quad + \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1}(F(s, \{g_1\}) - F(s, \{g_2\}))ds \Bigg| \\ &\le \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} \big|F(s, \{g_1\}) - F(s, \{g_2\})\big| ds \\ &\quad + \sum_{j=1}^{k} \big|\Upsilon_j(g_1(t_j^-)) - \Upsilon_j(g_2(t_j^-))\big| \\ &\le \frac{1}{(n-1)!}\int_{t_0}^{t} h(s)\big|g_1(s) - g_2(s)\big| ds + \sum_{j=1}^{k} M_j \big|g_1(t_j^-) - g_2(t_j^-)\big| \\ &\le \left(\frac{1}{(n-1)!}\int_{t_0}^{t_F} h(s)ds + \sum_{j=1}^{m} M_j\right) \|g_1 - g_2\|_{PC}. \end{aligned}$$

Then, Λ is contractive with respect to $\|\cdot\|_{PC}$. By virtue of Banach contraction principle, Λ is a Picard operator. The unique fixed point of this operator is the unique solution of (1.1) in $PC^n(I, R) \cap PC(I, R)$. This completes the proof. □

Remark 3.2. Theorem 3.1 includes [26, Theorem 4.1] in the case where $n = 1$ and $h(t) = h_0 > 0$.

Theorem 3.3. *Assume that* (H_2) *and* (H_3) *are satisfied. If F satisfies condition* (3.2), *then* (1.1) *has generalized Hyers–Ulam–Rassias stability on I with respect to (θ, μ).*

Proof. Let $y \in PC^n(I, R)$ be a solution to (2.2). For $i = 1, 2, \dots, n$, define a function by

$$x^{(n-i)}(t) = \begin{cases} \displaystyle\sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} + \frac{1}{(i-1)!}\int_{t_0}^{t} (t-s)^{i-1} F(s, \{x\})ds, & t \in [t_0, t_1], \\ \displaystyle\sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} + \Upsilon_1(x(t_1^-)) + \frac{1}{(i-1)!}\int_{t_0}^{t} (t-s)^{i-1} F(s, \{x\})ds, & t \in (t_1, t_2], \\ \displaystyle\sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} + \sum_{j=1}^{2}\Upsilon_j(x(t_j^-)) + \frac{1}{(i-1)!}\int_{t_0}^{t} (t-s)^{i-1} F(s, \{x\})ds, & t \in (t_2, t_3], \\ \cdot \\ \cdot \\ \cdot \\ \displaystyle\sum_{j=0}^{i-1} \frac{(t-t_0)^j y_{n-i+j}}{j!} + \sum_{j=1}^{m}\Upsilon_j(x(t_j^-)) + \frac{1}{(i-1)!}\int_{t_0}^{t} (t-s)^{i-1} F(s, \{x\})ds, & t \in (t_m, t_F]. \end{cases}$$

Then, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned} \left|y^{(n-i)}(t) - x^{(n-i)}(t)\right| &\le \left|y^{(n-i)}(t) - \sum_{j=0}^{i-1}\frac{(t-t_0)^j y_{n-i+j}}{j!} - \sum_{j=1}^{k}\Upsilon_j(y^{(n-i)}(t_j^-)) \right. \\ &\quad \left. - \frac{1}{(i-1)!}\int_{t_0}^{t}(t-s)^{i-1}F(s,\{y\})ds\right| \\ &\quad + \frac{1}{(i-1)!}\int_{t_0}^{t}(t-s)^{i-1}\left|F(s,\{y\}) - F(s,\{x\})\right|ds \\ &\quad + \sum_{j=1}^{k}\left|\Upsilon_j(y^{(n-i)}(t_j^-) - \Upsilon_j(x^{(n-i)}(t_j^-))\right| \\ &\le (m+\rho_\theta^n)(\theta(t)+\mu) + \frac{nS_0}{(i-1)!}\int_{t_0}^{t}(t-s)^{i-1}\left|y^{(n-i)}(s) - x^{(n-i)}(s)\right|ds \\ &\quad + \sum_{j=1}^{k} M_j\left|y^{(n-i)}(t_j^-) - x^{(n-i)}(t_j^-)\right|. \end{aligned}$$

Using Lemma 2.6, we deduce that, for $i = 1, 2, \dots, n$ and $t \in I$,

$$\left|y^{(n-i)}(t) - x^{(n-i)}(t)\right| \le (m+\rho_\theta^n)(\theta(t)+\mu)\prod_{t_0<t_k<t}(1+M_k)\exp\left(\frac{S_0(t-t_0)^n}{(n-1)!}\right) \le L_{F,m,\theta}(\theta(t)+\mu),$$

where

$$L_{F,m,\theta} = (m+\rho_\theta^n)\prod_{k=1}^{m}(1+M_k)\exp\left(\frac{S_0(t_F-t_0)^n}{(n-1)!}\right).$$

Hence, (1.1) is generalized Hyers–Ulam–Rassias stable on I with respect to (θ, μ). The proof is complete. □

Remark 3.4. Theorem 3.3 contains [26, Theorem 4.1] in the case when $n = 1$.

Remark 3.5. One can proceed as the same way to prove the Hyers–Ulam stability, generalized Hyers–Ulam stability, and Hyers–Ulam–Rassias stability of (1.1). The details are left to the reader.

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