General mixed width-integral of convex bodies

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Abstract

In this article, we introduce a new concept of general mixed width-integral of convex bodies, and establish some of its inequalities, such as isoperimetric inequality, Aleksandrov-Fenchel inequality, and cyclic inequality. We also consider the general width-integral of order $i$ and show its related properties and inequalities.

Keywords: General mixed width-integral, mixed width-integral, general width-integral of order $i$.

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1. Introduction and main results

Let $K^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we write $K^n_o$ and $K^n_c$, respectively. Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the $n$-dimensional volume of a body $K$. For the standard unit ball $B$ in $\mathbb{R}^n$, we use $\omega_n = V(B)$ to denote its volume.

If $K \in K^n$, then its support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, is defined by (see [6, 25])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

The study of width-integral has a long history. The notion of the classical width-integral was first considered by Blaschke (see [3]) and was further studied by Hardy, Littlewood and Pólya (see [12]). It was generalized to the mixed width-integral by Lutwak [19] in 1977. Many important results related to the mixed width-integral were obtained from these articles (see [13, 17, 18, 21]).
The mixed width-integral, $B(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n \in \mathcal{K}^n$ was defined by (see [19])
\begin{equation}
B(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} b(K_1, u) \cdots b(K_n, u) dS(u),
\end{equation}
where $dS(u)$ is the $(n-1)$-dimensional volume element on $S^{n-1}$ and $b(K, u)$ denotes the half width of $K$ in the direction $u$, namely, $b(K, u) = \frac{1}{2} h(K, u) + \frac{1}{2} h(K, -u)$. If there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$, then $K$ and $L$ are said to have similar width.

The main aim of this article is to define a corresponding notion of mixed width-integral, and to extend Lutwak’s inequalities to the entire family of this new mixed width-integral.

For $\tau \in (-1, 1)$, the general mixed width-integral, $B^{(\tau)}(K_1, \cdots, K_n)$, of $K_1, \cdots, K_n \in \mathcal{K}^n$ is defined by
\begin{equation}
B^{(\tau)}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_n, u) dS(u),
\end{equation}
where $b^{(\tau)}(K, u) = f_1(\tau) h(K, u) + f_2(\tau) h(K, -u)$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as follows
\begin{equation}
f_1(\tau) = \frac{(1 + \tau)^2}{2(1 + \tau^2)}, \quad f_2(\tau) = \frac{(1 - \tau)^2}{2(1 + \tau^2)}.
\end{equation}
Clearly,
\begin{align}
f_1(\tau) + f_2(\tau) &= 1, \quad f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau). \quad (1.4)
\end{align}

Together with (1.3), the case $\tau = 0$ in definition (1.2) is just Lutwak’s mixed width-integral $B(K_1, \cdots, K_n)$. Two convex bodies $K$ and $L$ are said to have similar general width if there exists a constant $\lambda > 0$ such that $b^{(\tau)}(K, u) = \lambda b^{(\tau)}(L, u)$ for all $u \in S^{n-1}$. If $b^{(\tau)}(K, u)b^{(\tau)}(L, u)$ is a constant for all $u \in S^{n-1}$, then we call $K$ and $L$ with joint constant general width.

The general operator belongs to the asymmetric Brunn-Minkowski theory which has its starting point in the theory of valuations in connection with isoperimetric and analytic inequalities (see [1, 2, 4, 5, 7–11, 14–16, 22, 24, 26, 30]).

The main results are the following: We first establish the isoperimetric and Aleksandrov-Fenchel inequalities for the general mixed width-integral.

**Theorem 1.1.** If $\tau \in (-1, 1)$ and $K_1, \cdots, K_n \in \mathcal{K}_n$, then
\begin{equation}
V(K_1) \cdots V(K_n) \leq B^{(\tau)}(K_1, \cdots, K_n)^n,
\end{equation}
with equality if and only if $K_1, \cdots, K_n$ are $n$-balls.

**Theorem 1.2.** If $\tau \in (-1, 1)$, $K_1, \cdots, K_n \in \mathcal{K}^n$ and $1 < m \leq n$, then
\begin{equation}
B^{(\tau)}(K_1, \cdots, K_n)^m \leq \prod_{i=1}^{m} B^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-m+1}, \cdots, K_{n-i+1}),
\end{equation}
with equality if and only if $K_{n-m+1}, \cdots, K_n$ are all of similar general width.

Moreover, we show a cyclic inequality for the general mixed width-integral.

**Theorem 1.3.** If $\tau \in (-1, 1)$ and $K, L \in \mathcal{K}^n$, then for $i < j < k$,
\begin{equation}
B^{(\tau)}_i(K, L)^{k-j} B^{(\tau)}_j(K, L)^{j-i} \geq B^{(\tau)}_j(K, L)^{k-j},
\end{equation}
with equality if and only if $K$ and $L$ have similar general width.

Here $B^{(\tau)}_i(K, L) = B^{(\tau)}_i(K, n-i; L, i)$ in which $K$ appears $n-i$ times and $L$ appears $i$ times.

The proofs of Theorems 1.1, 1.3 will be given in the Section 3 of this paper. In Section 4 we consider the general width-integral of order $i$ and establish its related properties and inequalities.
2. Preliminaries

The radial function, \( \rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty) \), of a compact star-shaped (about the origin) set \( K \) in \( \mathbb{R}^n \) is defined, for \( u \in S^{n-1} \), by (see [6, 25])

\[
\rho(K, u) = \max \{ \lambda \geq 0 : \lambda \cdot u \in K \}. \tag{2.1}
\]

The polar body, \( K^* \), of \( K \in \mathcal{K}^n \) is defined by (see [6, 25])

\[
K^* = \{ x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K \}. \tag{2.2}
\]

It is easy to check that for \( K \in \mathcal{K}^n \),

\[
(K^*)^* = K,
\]

and

\[
h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.
\]

An extension of the well-known Blaschke-Santaló inequality is as follows (see [20]):

**Theorem 2.1.** If \( K \in \mathcal{K}^n \), then

\[
V(K)V(K^*) \leq \omega_n^2,
\]

with equality if and only if \( K \) is an ellipsoid.

For \( K \in \mathcal{K}^n \) and \( i = 0, 1, \ldots, n-1 \), the quermassintegrals, \( W_i(K) \), of \( K \) is given by (see [6, 25])

\[
W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS_i(K, u), \tag{2.4}
\]

where \( S_i(K, \cdot) \) denotes the mixed surface area measure of \( K \). Besides, we know that

\[
W_0(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K, u) = V(K). \tag{2.5}
\]

The polar coordinate formula for volume of a body \( K \) in \( \mathbb{R}^n \) is

\[
V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u). \tag{2.6}
\]

3. Proofs of Theorems 1.1–1.3

**Proof of Theorem 1.1** It follows by Jensen’s inequality (see [12]) that

\[
B^{(\tau)}(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_n, u) dS(u) \geq n \omega_n^2 \left[ \int_{S^{n-1}} b^{(\tau)}(K_1, u)^{-1} \cdots b^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-1}, \tag{3.1}
\]

with equality if and only if \( K_1, \ldots, K_n \) have joint constant general width. Together with Hölder’s inequality (see [12]), we have

\[
\left[ \int_{S^{n-1}} b^{(\tau)}(K_1, u)^{-1} \cdots b^{(\tau)}(K_n, u)^{-1} dS(u) \right]^{-n} \geq \prod_{i=1}^n \left[ \int_{S^{n-1}} b^{(\tau)}(K_i, u)^{-n} dS(u) \right]^{-1}, \tag{3.2}
\]
with equality if and only if \( K_1, \cdots, K_n \) have similar general width. Using Minkowski’s inequality (see [12]), we have
\[
\frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(K, u)^{-n} dS(u) \leq \frac{1}{n} \int_{S^{n-1}} (f_1(\tau) h(K, u) + f_2(\tau) h(K, -u))^{-n} dS(u) \leq \frac{1}{n} V(K_i),
\]
with equality if and only if \( K_i \) is origin-symmetric. It follows from Theorem 2.1 that for inequality (3.3),
\[
\left[ \frac{1}{n^2} \int_{S^{n-1}} b^{(\tau)}(K, u)^{-n} dS(u) \right]^{\frac{1}{n}} = V(K_i),
\]
with equality if and only if \( K_i \) is an \( n \)-dimensional ellipsoid. From inequalities (3.1), (3.2) and (3.4), this yields
\[
V(K_1) \cdots V(K_n) \leq B^{(\tau)}(K_1, \cdots, K_n)^n.
\]

By the equality conditions of inequalities (3.1), (3.2) and (3.4), equality holds in (1.6) if and only if \( K_1, \cdots, K_n \) are \( n \)-balls.

**Lemma 3.1** ([17]). If \( f_0, f_1, \cdots, f_m \) are (strictly) positive continuous functions defined on \( S^{n-1} \) and \( \lambda_1, \cdots, \lambda_m \) are positive constants the sum of whose reciprocals is unity, then
\[
\int_{S^{n-1}} f_0(u) f_1(u) \cdots f_m(u) dS(u) \leq \prod_{i=1}^m \left[ \int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u) dS(u) \right]^{\frac{1}{m}},
\]
with equality if and only if there exist positive constants \( \alpha_1, \cdots, \alpha_m \) such that \( \alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u) \) for all \( u \in S^{n-1} \).

**Proof of Theorem 1.2** Let in Lemma 3.1
\[
\lambda_i = m \quad (1 \leq i \leq m),
\]
\[
f_0 = b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_{n-m}, u) \quad (f_0 = 1 \text{ if } m = n),
\]
\[
f_i = b^{(\tau)}(K_{n-i+1}, u) \quad (1 \leq i \leq m).
\]

Then
\[
\int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_n, u) dS(u)
\]
\[
\leq \prod_{i=1}^m \left[ \int_{S^{n-1}} b^{(\tau)}(K_1, u) \cdots b^{(\tau)}(K_{n-m}, u) b^{(\tau)}(K_{n-i+1}, u) dS(u) \right]^{\frac{1}{m}}.
\]

Combining with definition (1.2), we have
\[
B^{(\tau)}(K_1, \cdots, K_n)^m \leq \prod_{i=1}^m B^{(\tau)}(K_1, \cdots, K_{n-m}, K_{n-i+1}, \cdots, K_{n-i+1}).
\]
The equality condition of inequality (3.5) implies that equality holds in (1.7) if and only if \( K_{n-m+1}, \cdots, K_n \) are all of similar general width.
Proof of Theorem 1.3. It follows from Hölder’s inequality (see [12]) that
\[
B_i^{(\tau)}(K, L)^{\frac{k-j}{k-i}} B_j^{(\tau)}(K, L)^{\frac{j-i}{k-i}} = \left( \frac{1}{n} \int_{S_{n-1}} b(\tau)(K, u)^{n-i} b(\tau)(L, u)^j dS(u) \right)^{\frac{k-j}{k-i}} \times \left( \frac{1}{n} \int_{S_{n-1}} b(\tau)(K, u)^{n-k} b(\tau)(L, u)^k dS(u) \right)^{\frac{j-i}{k-i}} \geq \frac{1}{n} \int_{S_{n-1}} b(\tau)(K, u)^{n-j} b(\tau)(L, u)^j dS(u) = B_j^{(\tau)}(K, L).
\]
This gives
\[
B_i^{(\tau)}(K, L)^{k-j} B_j^{(\tau)}(K, L)^{j-i} \geq B_j^{(\tau)}(K, L)^{k-i}.
\]
The equality condition of Hölder’s inequality gets that equality holds in (1.8) if and only if \( K \) and \( L \) have similar general width.

Taking \( i = 0, j = i \) and \( k = n \) in inequality (1.8), we have

**Corollary 3.2.** If \( \tau \in (-1, 1) \) and \( K, L \in K^n \), then for \( 0 \leq i \leq n \),
\[
B_i^{(\tau)}(K, L)^n \leq B^{(\tau)}(K)^{n-i} B^{(\tau)}(L)^i, \tag{3.6}
\]
for \( i < 0 \) or \( i > n \), inequality (3.6) is reversed, with equality in every inequality if and only if \( i = n \) or, when \( i \neq n \), \( K \) and \( L \) have similar general width.

Let \( i = 1 \) and \( i = -1 \) in Corollary 3.2 respectively. The dual Minkowski type inequalities for the general mixed width-integral are as follows:

**Corollary 3.3.** If \( \tau \in (-1, 1) \) and \( K, L \in K^n \), then
\[
B_1^{(\tau)}(K, L)^n \leq B^{(\tau)}(K)^{n-1} B^{(\tau)}(L),
\]
with equality if and only if \( K \) and \( L \) have similar general width.

**Corollary 3.4.** If \( \tau \in (-1, 1) \) and \( K, L \in K^n \), then
\[
B_{-1}^{(\tau)}(K, L)^n \geq B^{(\tau)}(K)^{n+1} B^{(\tau)}(L)^{-1},
\]
with equality if and only if \( K \) and \( L \) have similar general width.

4. General width-integral of order \( i \)

In this section, we consider the general width-integral of order \( i \) and show its related properties and inequalities.

Taking \( K_1 = \cdots = K_{n-i} = K \) and \( K_{n-i+1} = \cdots = K_n = B \) in (1.2), the general width-integral of order \( i \), \( B_i^{(\tau)}(K) \), of \( K \in K^n \) is given by
\[
B_i^{(\tau)}(K) = \frac{1}{n} \int_{S_{n-1}} b^{(\tau)}(K, u)^{n-i} dS(u). \tag{4.1}
\]
Let \( K_1 = \cdots = K_n = K \) in (1.2). We write \( B^{(\tau)}(K) \) for \( B^{(\tau)}(K, \cdots, K) \) called the general width-integral of \( K \in K^n \).

If \( K_1, \cdots, K_m \in K^n \) and \( \lambda_1, \cdots, \lambda_m \in \mathbb{R} \), then the Minkowski linear combination is defined by (see [6, 25])
\[
\lambda_1 K_1 + \cdots + \lambda_m K_m = \{ \lambda_1 x_1 + \cdots + \lambda_m x_m : x_1 \in K_1, \cdots, x_m \in K_m \}.
\]
It is easy to verify that
\[ h(\lambda_1 K_1 + \cdots + \lambda_m K_m, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_m h(K_m, \cdot). \]

We now show that the general width-integral of \( \lambda_1 K_1 + \cdots + \lambda_m K_m \) is a homogeneous polynomial of degree \( n \) in \( \lambda_1, \cdots, \lambda_m \).

**Theorem 4.1.** Suppose \( \tau \in (-1, 1) \) and \( K_1, \cdots, K_m \in \mathcal{K}^n \). If \( K = \lambda_1 K_1 + \cdots + \lambda_m K_m \) then
\[
B^{(\tau)}(K) = \sum_{j_1=1}^{m} \cdots \sum_{j_n=1}^{m} \lambda_{j_1} \cdots \lambda_{j_n} B^{(\tau)}(K_{j_1}, \cdots, K_{j_n}). \tag{4.2}
\]

The following is a direct consequence of Theorem 4.1.

**Theorem 4.2.** Let \( \tau \in (-1, 1) \) and \( K \in \mathcal{K}^n \). If \( K_\mu = K + \mu B \) (\( \mu > 0 \)) then for \( j = 0, 1, \cdots, n, \)
\[
B_j^{(\tau)}(K_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} B_{j+i}^{(\tau)}(K) \mu^i. \tag{4.3}
\]

Further, we establish several inequalities for the general width-integral of order \( i \).

**Lemma 4.3.** If \( \tau \in (-1, 1) \) and \( K \in \mathcal{K}^n \), then
\[
B_{2n}^{(\tau)}(K) \leq V(K^*), \tag{4.4}
\]
with equality if and only if \( K \) is origin-symmetric.

**Proof.** Using Minkowski’s inequality (see [12]), we yield
\[
B_{2n}^{(\tau)}(K)^{-\frac{1}{n}} = \left[ \frac{1}{n} \int_{S_{n-1}} h^{(\tau)}(K, u)^{-n} dS(u) \right]^{-\frac{1}{n}} \\
= \left[ \frac{1}{n} \int_{S_{n-1}} (f_1(\tau) h(K, u) + f_2(\tau) h(K, -u))^{-n} dS(u) \right]^{-\frac{1}{n}} \\
\geq \left[ \frac{1}{n} \int_{S_{n-1}} (f_1(\tau) h(K, u))^{-n} dS(u) \right]^{-\frac{1}{n}} \\
+ \left[ \frac{1}{n} \int_{S_{n-1}} (f_2(\tau) h(K, -u))^{-n} dS(u) \right]^{-\frac{1}{n}} \\
= \left[ \frac{1}{n} \int_{S_{n-1}} h(K, u)^{-n} dS(u) \right]^{-\frac{1}{n}}.
\]

This implies
\[
B_{2n}^{(\tau)}(K) \leq \frac{1}{n} \int_{S_{n-1}} h(K, u)^{-n} dS(u) = V(K^*).
\]

The equality condition of Minkowski’s inequality gives that equality holds in (4.4) if and only if \( K \) and \(-K\) are dilated of one another, namely, \( K \) is origin-symmetric.

**Theorem 4.4.** If \( \tau \in (-1, 1) \) and \( K \in \mathcal{K}_c^n \), then for \( n < i < 2n, \)
\[
B_i^{(\tau)}(K) B_i^{(\tau)}(K^*) \leq \omega_n^2, \tag{4.5}
\]
For \( i < n \), inequality (4.5) is reversed, with equality in every inequality if and only if \( K \) is an ellipsoid centered at the origin.
Proof. Using Lemma 4.3 and Jensen’s inequality (see [12]), we have for \( i < 2n \) and \( i \neq n \)

\[
\frac{i-2n}{\omega_{n}^{(n-1)}} B^{(\tau)}_{i}(K)^{\frac{1}{n-1}} \geq B^{(\tau)}_{2n}(K)^{-\frac{1}{n}} \geq V(K^{*})^{-\frac{1}{n}}. \tag{4.6}
\]

Thus it follows from (4.6) that

\[
\frac{i-2n}{\omega_{n}^{(n-1)}} B^{(\tau)}_{i}(K^{*})^{\frac{1}{n-1}} \geq V(K)^{-\frac{1}{n}}. \tag{4.7}
\]

Together (4.6), (4.7) with Theorem 2.1, we get

\[
\left[ B^{(\tau)}_{i}(K) B^{(\tau)}_{i}(K^{*}) \right]^{\frac{1}{n-1}} \geq \omega_{n}^{\frac{2}{n-1}}. \tag{4.8}
\]

If \( n < i < 2n \) in inequality (4.8), then

\[
B^{(\tau)}_{i}(K) B^{(\tau)}_{i}(K^{*}) \leq \omega_{n}^{2}.
\]

If \( i < n \) in inequality (4.8), then

\[
B^{(\tau)}_{i}(K) B^{(\tau)}_{i}(K^{*}) \geq \omega_{n}^{2}.
\]

By the equality conditions of inequality (4.4), inequality (2.3) and Jensen’s inequality, we know that equality holds in every inequality if and only if \( K \) is an ellipsoid centered at the origin.

Lemma 4.5 ([6]). If \( K \in K^{n} \) and \( 0 \leq i < j < k \leq n \), then

\[
W_{j}(K)^{k-i} \geq W_{i}(K)^{k-j} W_{k}(K)^{j-i},
\]

with equality if and only if \( K \) is an \( n \)-ball.

Taking \( L = B \) in Theorem 1.3 the following is a direct result.

Lemma 4.6. For \( K \in K^{n} \) and \( \tau \in (-1,1) \), if \( i < j < k \) then

\[
B^{(\tau)}_{j}(K)^{k-i} \leq B^{(\tau)}_{i}(K)^{k-j} B^{(\tau)}_{k}(K)^{j-i},
\]

with equality if and only if \( K \) is of similar general width.

Lemma 4.7. If \( \tau \in (-1,1) \) and \( K \in K^{n} \), then

\[
B^{(\tau)}_{n-1}(K) = W_{n-1}(K).
\]

Proof. It follows by definition (4.1) that

\[
B^{(\tau)}_{n-1}(K) = \frac{1}{n} \int_{S^{n-1}} [f_{1}(\tau) h(K, u) + f_{2}(\tau) h(K, -u)] dS(u) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(u) = W_{n-1}(K).
\]

Theorem 4.8. For \( \tau \in (-1,1) \) and \( K \in K^{n} \), if \( i < n - 1 \) then

\[
W_{i}(K) \leq B^{(\tau)}_{i}(K), \tag{4.9}
\]

with equality if and only if \( K \) is an \( n \)-ball centered at the origin.
Proof. Using Lemma 4.5 it follows that
\[ W_i(K) \leq \omega_n^{i+1-n} W_{n-1}^{n-i}(K), \tag{4.10} \]
with equality if and only if \( K \) is an \( n \)-ball. By Lemma 4.6, we have
\[ \omega_n^{i+1-n} B_{n-1}^{(\tau)}(K)^{n-i} \leq B_{i}^{(\tau)}(K), \tag{4.11} \]
with equality if and only if \( K \) is of similar general width. Together \( (4.10), (4.11) \) with Lemma 4.7, this gives
\[ W_i(K) \leq B_{i}^{(\tau)}(K). \]

From the equality conditions of inequalities \( (4.10) \) and \( (4.11) \), we obtain that equality holds in \( (4.9) \) if and only if \( K \) is an \( n \)-ball centered at the origin.

Theorem 4.9. For \( \tau \in (-1, 1) \) and \( K \in \mathbb{K}^n \), if \( 0 < i < n \) then
\[ B_{n+i}^{(\tau)}(K) \leq W_{n-i}(K^*), \tag{4.12} \]
with equality if and only if \( K \) is an \( n \)-ball centered at the origin.

Proof. By Lemma 4.2, we get
\[ \omega_n^{n-i} V^i(K^*) \leq W_{n-i}(K^*), \tag{4.13} \]
with equality if and only if \( K^* \) is an \( n \)-ball. It follows from Lemma 4.6 that
\[ B_{n+i}^{(\tau)}(K)^n \leq \omega_n^{n-i} B_{2n}^{(\tau)}(K)^i, \tag{4.14} \]
with equality if and only if \( K \) is of similar general width. By \( (4.13), (4.14) \) and Lemma 4.3, we have
\[ B_{n+i}^{(\tau)}(K) \leq W_{n-i}(K^*). \]

The equality conditions of inequalities \( (4.13), (4.14) \) and \( (4.4) \) imply that equality holds in \( (4.12) \) if and only if \( K \) is an \( n \)-ball centered at the origin.

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