Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 4039–4051
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Strong convergence theorems for the generalized viscosity implicit rules of nonexpansive mappings in uniformly smooth Banach spaces

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Communicated by Y. J. Cho

Abstract

The aim of this paper is to introduce the generalized viscosity implicit rules of one nonexpansive mapping in uniformly smooth Banach spaces. Strong convergence theorems of the rules are proved under certain assumptions imposed on the parameters. As applications, we use our main results to solve fixed point problems of strict pseudocontractions in Hilbert spaces and variational inequality problems in Hilbert spaces. Finally, we also give one numerical example to support our main results.

Keywords: Fixed point, generalized implicit rules, generalized contraction, nonexpansive mapping, Banach spaces.
2010 MSC: 49H09, 47H10, 47H17, 49M05.

1. Introduction

In this paper, we assume that E is a real Banach space and E^* is the dual space of E. Let C be a subset of E and T be a self-mapping on C. Let $F(T)$ be the set of fixed points of mapping T.

A mapping $f : C \to C$ is called a contraction, if there exists a constant $\alpha \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \ \forall\ x, y \in C. \tag{1.1}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall\ x, y \in C. \tag{1.2}$$

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Received 2016-01-08

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the set of all positive integers and all positive real numbers, respectively. A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called to be an L-function if $\psi(0) = 0, \psi(t) > 0$ for all $t > 0$ and for every $s > 0$ there exists $u > s$ such that $\psi(t) \le s$ for each $t \in [s, u]$.

Let (E, d) be a metric space. A mapping $f : E \to E$ is said to be a (ψ, L)-contraction if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an L-function and $d(f(x), f(y)) < \psi(d(x, y))$, for all $x, y \in E, x \ne y$. A mapping $f : E \to E$ is said to be a Meir-Keeler type mapping if for each $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for each $x, y \in E$, with $\epsilon \le d(x, y) < \epsilon + \delta$, we have $d(f(x), f(y)) < \epsilon$.

Proposition 1.1 ([10])**.** *Let (E, d) be a metric space and $f : E \to E$ be a mapping. The following assertions are equivalent:*

(i) *f is a Meir-Keeler type mapping;*

(ii) *there exists an L-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that f is a (ψ, L)-contraction.*

Proposition 1.2 ([16])**.** *Let C be a convex subset of a Banach space E and let $f : C \to C$ be a Meir-Keeler type mapping. Then, for each $\epsilon > 0$ there exists $r \in (0, 1)$ such that*

$$\|x - y\| \ge \epsilon \text{ implies } \|f(x) - f(y)\| \le r \|x - y\| .$$

In what follows, a Meir-Keeler type mapping or (ψ, L)-contraction is called a generalized contraction mapping. We assume that the L-function from the definition of (ψ, L)-contraction is continuous, strictly increasing and $\lim_{t\to\infty} \eta(t) = \infty$, where $\eta(t) = t - \psi(t)$ for all $t \in \mathbb{R}^+$.

Fixed Point Theory plays a very important role for solving all kinds of problems, such as variational inequality problems in Hilbert spaces or Banach spaces, equilibrium problems, optimization problems and so on. Recently, viscosity iterative algorithms for approximating a fixed point of nonexpansive mappings have been investigated extensively by many authors, see [4, 6, 7, 9, 11, 12, 14, 15, 17] and the references therein. For instance, Xu [17] introduced an explicit viscosity method for nonexpansive mappings in Hilbert spaces and uniformly smooth Banach spaces. Strong convergence theorems are obtained under some suitable conditions on parameters. Song et al.[14] studied a viscosity algorithm for a family of nonexpansive mappings in a real strictly convex Banach space with a uniformly Gâteaux differentiable norm by using uniformly asymptotically regular condition.

Very recently, iterative sequence for the implicit midpoint rule has been studied by many authors, because it is a powerful method for solving ordinary differential equations; see [1, 2, 5, 8, 13, 18, 19] and the references therein. Recently, Xu et al. [18] considered the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \ n \ge 0. \tag{1.3}$$

They proved that the iterative sequence defined by (1.3) converges strongly to a fixed point of T which also solves the following variational inequality in Hilbert spaces:

$$\langle (I - f)q, x - q\rangle \ge 0, \ x \in F(T). \tag{1.4}$$

Very recently, Ke et al.[8] applied the viscosity technique to the implicit rules of nonexpansive mappings in Hilbert spaces. More precisely, they proposed the following two viscosity implicit rules:

$$x_{n+1} = \alpha_n Q(x_n) + (1 - \alpha_n)T(s_n x_n + (1 - s_n)x_{n+1}), \tag{1.5}$$

and

$$x_{n+1} = \alpha_n x_n + \beta_n Q(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}). \tag{1.6}$$

They obtained that the sequence $\{x_n\}$ generated by (1.5) and (1.6) converges strongly to a fixed point of nonexpansive mapping T, which also solves variational inequality (1.4). The following questions naturally arise:

Question 1. In ke et al.[8], Step 5 in the proof of Theorem 3.1 and Theorem 3.2 is complicated. Can we use techniques to simplify the step 5?

Question 2. Can we extend the main results of Ke et al.[8] from Hilbert spaces to a general Banach spaces? such as uniformly smooth Banach spaces.

Question 3. Can we replace strict contractions by more generalized contractions? Such as Meir-Keeler type mappings or a (ψ, L)-functions.

The aim of this paper is to give affirmative answer to these questions mentioned above. We study the generalized viscosity implicit rules (1.6) of one nonexpansive mapping in uniformly smooth Banach spaces. We prove some strong convergence theorems for finding a fixed point of one nonexpansive mapping under suitable assumptions imposed on the parameters. As applications, we apply our main results to solve fixed point problems of strict pseudocontractions in Banach spaces and variational inequality problems in Hilbert spaces. Finally, we give some numerical examples for supporting our main results.

2. Preliminaries

The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \ \|x^*\| = \|x\| \right\}, \forall\, x \in E.$$

It is well known that if E is a Hilbert space, then J is the identity mapping and if E is smooth, then J is single-valued, which is denoted by j.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in S(E), \ \|y\| \le t \right\}.$$

A Banach space E is said to be uniformly smooth if $\dfrac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Furthermore, Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. Typical example of uniformly smooth Banach spaces is L^p, where $p > 1$. Precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. It is well known that, if E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth.

The following lemmas are very useful for proving our main results.

Lemma 2.1 ([17]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \alpha_n)a_n + \delta_n, \ n \ge 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([15]). *Let C be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \ne \emptyset$ and $f : C \to C$ be a generalized contraction mapping. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0, 1)$, converges strongly to $\hat{x} \in F(T)$, which solves the variational inequality:*

$$\langle f(\hat{x}) - \hat{x}, j(z - \hat{x}) \rangle \le 0, \forall\, z \in F(T).$$

Lemma 2.3 ([15]). *Let C be a nonempty closed and convex subset of a uniformly smooth Banach space E. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \ne \emptyset$ and $f : C \to C$ be a generalized contraction mapping. Assume that $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0, 1)$, converges strongly to $\hat{x} \in F(T)$ as $t \to 0$. Suppose that $\{x_n\}$ is bounded sequence such that $x_n - Tx_n \to 0$ as $n \to \infty$. Then*

$$\limsup_{n\to\infty} \langle f(\hat{x}) - \hat{x}, j(x_n - \hat{x}) \rangle \le 0.$$

3. Main results

Theorem 3.1. *Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)x^*, j(y - x^*)\rangle \geq 0, \ \text{for all } y \in F(T).$$

Proof. First, we show that $\{x_n\}$ is bounded. Indeed, take $p \in F(T)$ arbitrarily, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}) - p\| \\
&= \|\alpha_n(x_n - p) + \beta_n(f(x_n) - p) + \gamma_n(T(s_n x_n + (1 - s_n)x_{n+1}) - p)\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - p\| + \gamma_n\|T(s_n x_n + (1 - s_n)x_{n+1}) - p\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + \gamma_n\|s_n x_n + (1 - s_n)x_{n+1} - p\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\psi(\|x_n - p\|) + \beta_n\|f(p) - p\| + \gamma_n\|s_n(x_n - p) + (1 - s_n)(x_{n+1} - p)\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n\psi(\|x_n - p\|) + \beta_n\|f(p) - p\| + \gamma_n s_n\|x_n - p\| + \gamma_n(1 - s_n)\|x_{n+1} - p\|.
\end{aligned}$$

It follows that

$$(1 - \gamma_n(1 - s_n))\|x_{n+1} - p\| \leq (\alpha_n + \gamma_n s_n + \beta_n\psi)\|x_n - p\| + \beta_n\|f(p) - p\|,$$

that is

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{\alpha_n + \gamma_n s_n + \beta_n\psi}{1 - \gamma_n(1 - s_n)}\|x_n - p\| + \frac{\beta_n}{1 - \gamma_n(1 - s_n)}\|f(p) - p\| \\
&= \left(1 - \frac{\beta_n\eta}{1 - \gamma_n(1 - s_n)}\right)\|x_n - p\| + \frac{\beta_n\eta}{1 - \gamma_n(1 - s_n)} \cdot \eta^{-1}\|f(p) - p\|.
\end{aligned}$$

Thus, we have

$$\|x_{n+1} - p\| \leq \max\{\|x_n - p\|, \eta^{-1}\|f(p) - p\|\}.$$

By induction, we obtain

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \eta^{-1}\|f(p) - p\|\}.$$

Hence we obtain that $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Set $y_n = \frac{x_{n+1}-\alpha_n x_n}{1-\alpha_n}$ for all $n \geq 0$. We observe

$$\begin{aligned}
y_{n+1} - y_n &= \frac{x_{n+2} - \alpha_{n+1}x_{n+1}}{1-\alpha_{n+1}} - \frac{x_{n+1} - \alpha_n x_n}{1-\alpha_n} \\
&= \frac{\beta_{n+1} f(x_{n+1}) + \gamma_{n+1}T(s_{n+1}x_{n+1} + (1-s_{n+1})x_{n+2})}{1-\alpha_{n+1}} - \frac{\beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1})}{1-\alpha_n} \\
&= \frac{\beta_{n+1}}{1-\alpha_{n+1}}(f(x_{n+1}) - f(x_n)) + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}[T(s_{n+1}x_{n+1} + (1-s_{n+1})x_{n+2}) \\
&\quad - T(s_n x_n + (1-s_n)x_{n+1})] + (\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n})(f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq &\frac{\beta_{n+1}}{1-\alpha_{n+1}}\psi(\|x_{n+1} - x_n\|) + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}(1-s_{n+1})\|x_{n+2} - x_{n+1}\| \\
&+ \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}s_n\|x_{n+1} - x_n\| + |\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}| \\
&\times \|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|.
\end{aligned} \tag{3.2}$$

However,

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|\alpha_{n+1}(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)(x_n - T(s_n x_n + (1-s_n)x_{n+1})) \\
&\quad + \beta_{n+1}[f(x_{n+1}) - f(x_n)] + (\beta_{n+1} - \beta_n)\cdot(f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})) \\
&\quad + \gamma_{n+1}[T(s_{n+1}x_{n+1} + (1-s_{n+1})x_{n+2}) - T(s_n x_n + (1-s_n)x_{n+1})]\| \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\| \\
&\quad + \beta_{n+1}\psi(\|x_{n+1} - x_n\|) + |\beta_{n+1} - \beta_n|\cdot\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\| \\
&\quad + \gamma_{n+1}(1-s_{n+1})\|x_{n+2} - x_{n+1}\| + \gamma_{n+1}s_n\|x_{n+1} - x_n\| \\
&= (\alpha_{n+1} + \gamma_{n+1}s_n + \beta_{n+1}\psi)\|x_{n+1} - x_n\| + \gamma_{n+1}(1-s_{n+1})\|x_{n+2} - x_{n+1}\| \\
&\quad + |\alpha_{n+1} - \alpha_n|\cdot\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\| \\
&\quad + |\beta_{n+1} - \beta_n|\cdot\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|.
\end{aligned} \tag{3.3}$$

It follows that

$$\begin{aligned}
&\|x_{n+2} - x_{n+1}\| \\
&\leq \frac{\alpha_{n+1} + \gamma_{n+1}s_n + \beta_{n+1}\psi}{1-\gamma_{n+1}(1-s_{n+1})}\|x_{n+1} - x_n\| + \frac{|\alpha_{n+1} - \alpha_n|}{1-\gamma_{n+1}(1-s_{n+1})}\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\| \\
&\quad + \frac{|\beta_{n+1} - \beta_n|}{1-\gamma_{n+1}(1-s_{n+1})}\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\| \\
&= \left[1 - \frac{\beta_{n+1}\eta + \gamma_{n+1}(s_{n+1} - s_n)}{1-\gamma_{n+1}(1-s_{n+1})}\right]\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\frac{\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|}{1-\gamma_{n+1}(1-s_{n+1})} \\
&\quad + |\beta_{n+1} - \beta_n|\frac{\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|}{1-\gamma_{n+1}(1-s_{n+1})}.
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2), we have

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\leq \frac{\beta_{n+1}}{1-\alpha_{n+1}}\psi(\|x_{n+1} - x_n\|) + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}\cdot s_n\|x_{n+1} - x_n\| + |\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}|
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\| + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}(1-s_{n+1})\\
&\quad \times [(1 - \frac{\beta_{n+1}\eta + \gamma_{n+1}(s_{n+1}-s_n)}{1-\gamma_{n+1}(1-s_{n+1})})\|x_{n+1}-x_n\| + |\alpha_{n+1}-\alpha_n|\frac{\|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|}{1-\gamma_{n+1}(1-s_{n+1})}\\
&\quad + |\beta_{n+1}-\beta_n|\frac{\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|}{1-\gamma_{n+1}(1-s_{n+1})}]\\
&= \frac{1}{1-\alpha_{n+1}}\{\beta_{n+1}\psi + (1-\alpha_{n+1}-\beta_{n+1})s_n + (1-\alpha_{n+1}-\beta_{n+1})(1-s_{n+1})\\
&\quad \times (1 - \frac{\beta_{n+1}\eta + \gamma_{n+1}(s_{n+1}-s_n)}{1-\gamma_{n+1}(1-s_{n+1})})\} \cdot \|x_{n+1}-x_n\| + (|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}|\\
&\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}(1-s_{n+1})\frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-s_{n+1})})\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\|\\
&\quad + \frac{1-\alpha_{n+1}-\beta_{n+1}}{1-\alpha_{n+1}}(1-s_{n+1})\frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-s_{n+1})} \cdot \|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|\\
&\le (1 - \frac{\eta\beta_{n+1}}{1-\alpha_{n+1}})\|x_{n+1}-x_n\| + [|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}| + \frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-s_{n+1})}]\\
&\quad \times \|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\| + \frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-s_{n+1})} \cdot \|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|\\
&\le (1 - \frac{\eta\beta_{n+1}}{1-\alpha_{n+1}})\|x_{n+1}-x_n\| + [|\frac{\beta_{n+1}}{1-\alpha_{n+1}} - \frac{\beta_n}{1-\alpha_n}| + \frac{|\beta_{n+1}-\beta_n|}{1-\gamma_{n+1}(1-s_{n+1})}\\
&\quad + \frac{|\alpha_{n+1}-\alpha_n|}{1-\gamma_{n+1}(1-s_{n+1})}]M_1,
\end{aligned}
$$

where $M = \sup_{n\ge 0}\{\|f(x_n) - T(s_n x_n + (1-s_n)x_{n+1})\| + \|x_n - T(s_n x_n + (1-s_n)x_{n+1})\|\}$.

Hence, we have

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \le 0.$$

It follows that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. By the definition of $\{y_n\}$, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.5}$$

Next, we prove that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In fact, we observe

$$
\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\|\\
&\le \|x_n - x_{n+1}\| + \alpha_n\|x_n - Tx_n\| + \beta_n\|f(x_n) - Tx_n\| + \gamma_n\|s_n x_n + (1-s_n)x_{n+1} - x_n\|\\
&\le \|x_n - x_{n+1}\| + \alpha_n\|x_n - Tx_n\| + \beta_n\|f(x_n) - Tx_n\| + \gamma_n(1-s_n)\|x_{n+1}-x_n\|,
\end{aligned}
$$

which implies

$$\|x_n - Tx_n\| \le \frac{1+\gamma_n(1-s_n)}{1-\alpha_n}\|x_{n+1}-x_n\| + \frac{\beta_n}{1-\alpha_n}\|f(x_n) - Tx_n\|.$$

Then by (3.5) and condition (iii), we get

$$\|x_n - Tx_n\| \to 0 \text{ as } n\to\infty. \tag{3.6}$$

Let $\{x_t\}$ be a sequence defined by $x_t = tf(x_t) + (1-t)Tx_t$, by Lemma 2.2, we have that $\{x_t\}$ converges strongly to a fixed point x^* of T, which solves the variational inequality:

$$\langle (I-f)x^*, j(x-x^*)\rangle \ge 0,\ x\in F(T).$$

It follows from (3.6) and Lemma 2.3 that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, j(x_n - x^*)\rangle \le 0. \tag{3.7}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. Assume that the sequence $\{x_n\}$ does not converge strongly to $x^* \in F(T)$. Then there exists $\epsilon > 0$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - x^*\| \geq \epsilon$, for all $j \in \{0, 1 \cdots\}$. For this ϵ there exists $r \in (0,1)$ such that

$$\|f(x_{n_j}) - f(x^*)\| \leq r\|x_{n_j} - x^*\|.$$

Then we have

$$\begin{aligned}
&\|x_{n_{j+1}} - x^*\|^2 \\
&\quad= \langle \alpha_{n_j} x_{n_j} + \beta_{n_j} f(x_{n_j}) + \gamma_{n_j} T(s_{n_j} x_{n_j} + (1 - s_{n_j}) x_{n_j+1}) - x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad= \langle \alpha_{n_j} x_{n_j} + \beta_{n_j} f(x_{n_j}) + \gamma_{n_j} T(s_{n_j} x_{n_j} + (1 - s_{n_j}) x_{n_j+1}) - (\alpha_{n_j} + \beta_{n_j} + \gamma_{n_j}) x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad= \alpha_{n_j} \langle x_{n_j} - x^*, j(x_{n_j+1} - x^*)\rangle + \beta_{n_j} \langle f(x_{n_j}) - f(x^*), j(x_{n_j+1} - x^*)\rangle \\
&\qquad+ \beta_{n_j} \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle + \gamma_{n_j} \langle T(s_{n_j} x_{n_j} + (1 - s_{n_j}) x_{n_j+1}) - x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad\leq \alpha_{n_j} \|x_{n_j} - x^*\| \|x_{n_j+1} - x^*\| + r\beta_{n_j} \|x_{n_j} - x^*\| \|x_{n_j+1} - x^*\| + \gamma_{n_j} s_{n_j} \|x_{n_j} - x^*\| \|x_{n_j+1} - x^*\| \\
&\qquad+ \gamma_{n_j} (1 - s_{n_j}) \|x_{n_j+1} - x^*\|^2 + \beta_{n_j} \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad= (\alpha_{n_j} + r\beta_{n_j} + \gamma_{n_j} s_{n_j}) \|x_{n_j} - x^*\| \|x_{n_j+1} - x^*\| + \gamma_{n_j} (1 - s_{n_j}) \|x_{n_j+1} - x^*\|^2 \\
&\qquad+ \beta_{n_j} \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad\leq \frac{\alpha_{n_j} + r\beta_{n_j} + \gamma_{n_j} s_{n_j}}{2} \|x_{n_j} - x^*\|^2 + \frac{\alpha_{n_j} + r\beta_{n_j} + \gamma_{n_j} s_{n_j}}{2} \|x_{n_{j+1}} - x^*\|^2 \\
&\qquad+ \gamma_{n_j} (1 - s_{n_j}) \|x_{n_j+1} - x^*\|^2 + \beta_{n_j} \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
&\|x_{n_{j+1}} - x^*\|^2 \\
&\quad\leq \frac{\alpha_{n_j} + r\beta_{n_j} + \gamma_{n_j} s_{n_j}}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}} \|x_{n_j} - x^*\|^2 + \frac{2\beta_{n_j}}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}} \\
&\qquad\times \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle \\
&\quad= \left(1 - \frac{2 - 2\alpha_{n_j} - 2r\beta_{n_j} - 2\gamma_{n_j}}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}}\right) \|x_{n_j} - x^*\|^2 + \frac{2 - 2\alpha_{n_j} - 2r\beta_{n_j} - 2\gamma_{n_j}}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}} \\
&\qquad\times \frac{2\beta_{n_j}}{2 - 2\alpha_{n_j} - 2r\beta_{n_j} - 2\gamma_{n_j}} \cdot \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
\alpha'_{n_j} &= \frac{2 - 2\alpha_{n_j} - 2r\beta_{n_j} - 2\gamma_{n_j}}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}} \\
&= \frac{2\beta_{n_j}(1 - r)}{2 - \alpha_{n_j} - r\beta_{n_j} + \gamma_{n_j} s_{n_j} - 2\gamma_{n_j}} \\
&= \frac{2\beta_{n_j}(1 - r)}{1 + \beta_{n_j}(1 - r) + \gamma_{n_j}(s_{n_j} - 1)} \subset [0, 1].
\end{aligned}$$

We notice

$$\frac{2\beta_{n_j}(1 - r)}{1 + \beta_{n_j}(1 - r) + \gamma_{n_j}(s_{n_j} - 1)} > \frac{2\beta_{n_j}(1 - r)}{1 + \beta_{n_j}(1 - r)} > \beta_{n_j}(1 - r).$$

As $\sum_{n=0}^{\infty} \beta_{n_j} = \infty$, so we have $\sum_{n=0}^{\infty} \alpha'_{n_j} = \infty$. Let

$$\sigma'_{n_j} = \frac{2\beta_{n_j}}{2 - 2\alpha_{n_j} - 2r\beta_{n_j} - 2\gamma_{n_j}} \cdot \langle f(x^*) - x^*, j(x_{n_j+1} - x^*)\rangle.$$

Then it follows from (3.7) that $\limsup_{n\to\infty} \sigma'_{n_j} \leq 0$. So we obtain that $x_{n_j} \to x^*$ as $j \to \infty$. The contradiction permits us to conclude that $\{x_n\}$ converges strongly to $x^* \in F(T)$. This finishes the proof. □

The following results can be obtained by Theorem 3.1 easily. We omit the details.

Theorem 3.2. *Let E be a uniformly smooth Banach space, C a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\frac{x_n + x_{n+1}}{2}), \tag{3.8}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)x^*, j(y - x^*)\rangle \geq 0, \forall\ y \in F(T).$$

Corollary 3.3. *Let C be a nonempty closed convex subset of Hilbert space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1 - s_n)x_{n+1}), \tag{3.8}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $0 < \varepsilon \leq s_n \leq s_{n+1} < 1$ *for all* $n \geq 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)x^*, y - x^*\rangle \geq 0, \forall\ y \in F(T).$$

Corollary 3.4. *Let C be a nonempty closed convex subset of Hilbert space E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(\frac{x_n + x_{n+1}}{2}), \tag{3.10}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)x^*, y - x^*\rangle \geq 0, \forall\ y \in F(T).$$

Remark 3.5. Theorem 3.1 improves and extends Theorem 3.2 of Ke and Ma[8] in the following aspects.

(1) Strict contraction is replaced by a generalized contraction.

(2) From Hilbert spaces to more general uniformly smooth Banach spaces.

(3) Condition $\lim_{n\to\infty}\gamma_n = 1$ is removed and condition $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ is weakened as $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0$.

(4) Our proof of main results are very different from ones in Ke and Ma[8]. Precisely, we use other method to deal with the proof of step 2 and step 5, in this way, we simplify the proof of main results.

4. Applications

(I) Application to variational inequality problems in Hilbert spaces.

Let C be a nonempty closed convex subset of a Hilbert space H. Recall the following definitions.

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in C.$$

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,\ \forall x, y \in C.$$

Let $A : C \to H$ be a nonlinear operator. The classical variational inequality is to find x^* satisfying

$$\langle Ax^*, x - x^*\rangle \geq 0,\ \forall\ x \in C. \tag{4.1}$$

We use VI(A, C) to denoted the set of solutions of (4.1).

Ceng et al. [3] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0,\ \forall\, x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0,\ \forall\, x \in C, \end{cases} \tag{4.2}$$

which is called a general system of variational inequalities, where $A,\ B : C \to H$ are two nonlinear mappings, $\lambda > 0$ and $\mu > 0$ are two constants. They studied the following algorithm: $x_1 = u \in C$ and

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda Ay_n). \end{cases} \tag{4.3}$$

By using a relaxed extragradient method, they proved some strong convergence theorems under appropriate conditions in a real Hilbert space.

Lemma 4.1 ([3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (4.2) *if and only if x^* is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C(P_C(x - \mu Bx) - \lambda AP_C(x - \mu Bx)), \forall\ x \in C,$$

where $y^* = Q_C(x^* - \mu Bx^*)$.

Theorem 4.2. *Let C be a nonempty closed convex subset of Hilbert space H. Let the mappings $A, B : C \to H$ be α-inverse-strongly monotone and β-inverse-strongly monotone with $F(G) \neq \emptyset$, where $G : C \to C$ is a mapping defined by Lemma 4.1. Let $f : C \to C$ be a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n y_n, \\ y_n = Q_C(u_n - \lambda A u_n), \\ u_n = Q_C(z_n - \mu B z_n), \\ z_n = s_n x_n + (1 - s_n) x_{n+1}, \end{cases} \tag{4.4}$$

where $\lambda \in (0, 2\alpha), \mu \in (0, 2\beta)$. Let $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $0 < \varepsilon \le s_n \le s_{n+1} < 1$ *for all* $n \ge 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* G, which is also the solution of the variational inequality

$$\langle (I - f)x^*, y - x^* \rangle \ge 0, \forall\ y \in F(G),$$

and (x^*, y^*) is a solution of problem (4.2), where $y^* = Q_C(x^* - \mu B x^*)$.

Proof. By Remark 2.1 of [3], we know that G is nonexpansive. So we obtain the desired results by Theorem 3.1 and Lemma 4.2. □

(II) Application to strict pseudocontractive mappings.

Let K be a nonempty subset of a Hilbert space H. Recall that a mapping $T : K \to H$ is said to be k-strict pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2, \forall\ x, y \in K. \tag{4.5}$$

Lemma 4.3 ([20]). *Let H be a Hilbert space, K be a closed convex subset of H. If T is a k-strict pseudocontractive mapping on K, then the fixed point set $F(T)$ is closed convex, so that the projection $P_{F(T)}$ is well defined.*

Lemma 4.4 ([20]). *Let H be a Hilbert space, K be a closed convex subset of H. If $T : K \to H$ is a k-strict pseudocontractive mapping with $F(T) \neq \emptyset$, then $F(P_K T) = F(T)$.*

Lemma 4.5 ([20]). *Let H be a Hilbert space, K be a closed convex subset of H. If $T : K \to H$ is a k-strict pseudocontractive mapping. Define a mapping $S : K \to K$ by $Sx = \lambda x + (1 - \lambda)Tx$ for all $x \in K$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.*

Theorem 4.6. *Let C be a nonempty closed convex subset of Hilbert space E. Let $T : C \to H$ be a k-strict pseudocontractive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a generalized contraction mapping. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n P_C S(s_n x_n + (1 - s_n) x_{n+1}), \tag{4.6}$$

where $S : C \to H$ is defined by $Sx = \delta x + (1 - \delta)Tx, \forall\ x \in C$, $\delta \in [k, 1)$. Let $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;

(ii) $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $0 < \varepsilon \le s_n \le s_{n+1} < 1$ *for all* $n \ge 0$.

Then $\{x_n\}$ *converges strongly to a fixed point* x^* *of the nonexpansive mapping* T*, which is also the solution of the variational inequality*

$$\langle (I-f)x^*, y - x^* \rangle \ge 0, \forall\, y \in F(T).$$

Proof. By Lemma 4.4 and 4.5, we have that $P_C S$ is nonexpansive and $F(P_C S) = F(T)$. So we obtain the desired results by Theorem 3.1 immediately. $\square$

5. Numerical Examples

Example 5.1. Let inner product $< \cdot, \cdot >: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3,$$

and the usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$\|\mathbf{x}\| = \sqrt{x_1^2 + y_1^2 + z_1^2},\ \forall\ \mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Let $T, f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T\mathbf{x} = f(\mathbf{x}) = \frac{1}{4}\mathbf{x},\ \forall\ x \in \mathbb{R}$. Let

$$\alpha_n = \frac{1}{4} + \frac{1}{4n},\ \beta_n = \frac{1}{4n},\ \gamma_n = \frac{3}{4} + \frac{1}{2n},\ s_n = \frac{1}{4}, \forall n \in \mathbb{N}.$$

Let $\{x_n\}$ be a sequence generated by (3.8). It is easy to see that $F(T) = \{0\}$. Then $\{\mathbf{x}_n\}$ converges strongly to 0 by Corollary 3.3.

We can rewrite (3.8) as follows:

$$\mathbf{x}_{n+1} = \frac{19n + 18}{55n + 6}\mathbf{x}_n. \tag{5.1}$$

Choosing $\mathbf{x}_1 = (1, 2, 3)$ in (5.1), we have the following numerical results in Figure 1 and Figure 2.

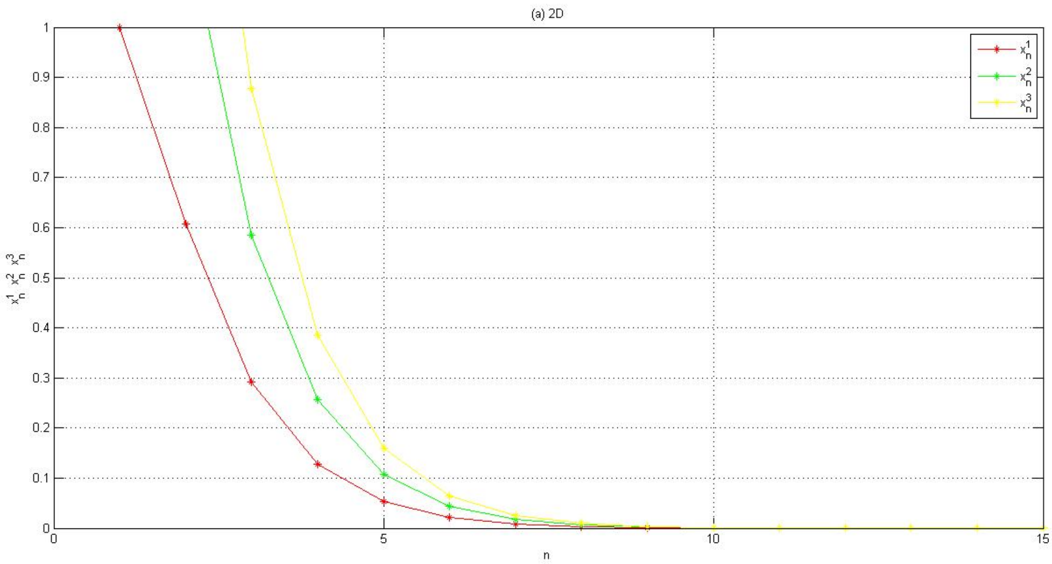

Figure 1

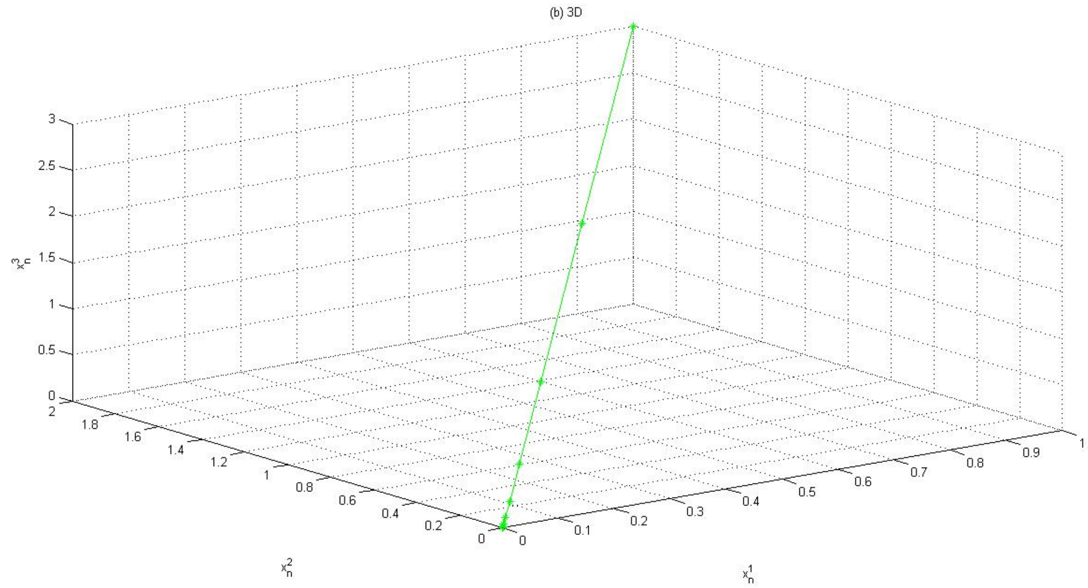

Figure 2

Acknowledgment

This work was supported by the NSF of China (No. 11401063), the Natural Science Foundation of Chongqing (cstc2014jcyjA00016), Science and Technology Project of Chongqing Education Committee (Grant No. KJ1500314) and the graduate students' innovative research project of Chongqing normal University (YKC16001).

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