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The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems

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Abstract

In this paper we study a class of operator equations $A(x,x) + B(x,x) = x$ in ordered Banach spaces, where A, B are two mixed monotone operators. Various theorems are established to guarantee the existence of a unique solution to the problem. In addition, associated iterative schemes have been established for finding the approximate solution converging to the fixed point of the problem. We also study the solution of the nonlinear eigenvalue equation $A(x,x) + B(x,x) = \lambda x$ and discuss its dependency to the parameter. Our results extend and improve many known results in this field of study. We have also successfully demonstrated the application of our results to the study of nonlinear fractional differential equations with two-point boundary conditions.

Keywords: Mixed monotone operator, hypo-homogeneous mixed monotone operator, existence and uniqueness, fractional differential equation.
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1. Introduction

Mixed monotone operators were introduced by Guo and Lakshmikantham in [9]. Thereafter, many authors have investigated various kinds of nonlinear mixed monotone operators in Banach spaces such as nonlinear operators with concave-convex (see [4]), mixed monotone operators with α-concave-convex (see [33, 34]), nonlinear operators with ϕ-concave-convex (see [10, 26]), nonlinear operators with e-concave-convex

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(see [39]), and also obtained a lot of important results on mixed monotone operators (see [1, 2, 7, 8, 11, 13, 16, 17, 21, 22, 23, 24, 27, 28, 29, 31, 32, 38]). These studies not only have theoretical significance but also have a wide range of applications in engineering, nuclear physics, biology, chemistry, technology, etc. Because of the crucial role played by nonlinear equations in applied science as well as mathematics, nonlinear functional analysis has been an active area of research, and nonlinear operators with connection to nonlinear (fractional) differential and integral equations have been extensively studied over the past several decades (see [5, 6, 12, 14, 15, 19, 25, 29, 30, 31, 32, 35, 36, 37]).

In [7], Guo studied the existence and uniqueness of positive solutions to the following operator equation on ordered Banach spaces E

$$A(x,x)=x,\ \ x\in P,$$

where P is a cone in E and $A:P\times P\to E$ is an α-concave mixed monotone operator.

In [27], Zhai and Anderson considered the existence and uniqueness of positive solutions to the following operator equation in ordered Banach spaces E

$$Ax+Bx+Cx=x,\ \ x\in P,$$

where P is a cone in E and $A:P\to E$ is an increasing α-concave operator, $B:P\to E$ is an increasing hypo-homogeneous operator and $C:P\to E$ is an homogeneous operator.

In [28], using the fixed point theorem of mixed monotone operators, Zhai and Hao studied the existence and uniqueness of positive solutions to the following operator equation in Banach spaces E

$$A(x,x)+Bx=x,\ \ x\in P,$$

where P is a cone in E and $A:P\times P\to E$ is an α-concave mixed monotone operator, $B:P\to E$ is an increasing hypo-homogeneous operator.

In [21], Wang and Zhang studied a class of sum operator equations

$$Ax+Bx+C(x,x)=x,\ \ x\in P$$

on a cone P of a Banach space E, where $A:P\to E$ is an increasing sub-homogeneous operator, $B:P\to E$ is a decreasing operator, $C:P\times P\to E$ is a mixed monotone operator and satisfies the following conditions:

$$B(t^{-1}y)\geq tBy,\quad C(tx,t^{-1}y)\geq t^{\alpha}C(x,y),\quad \text{for all } t\in(0,1),x,y\in P.$$

By using the properties of cone and fixed point theorems for mixed monotone operators, the existence and uniqueness of a positive solution are obtained.

Motivated by the above work, this paper considers the existence and uniqueness of positive solutions to the following operator equation in ordered Banach spaces E

$$A(x,x)+B(x,x)=x,\ \ x\in P, \tag{1.1}$$

where P is a cone in E, and $A,B:P\times P\to P$ are two mixed monotone operators, which satisfy the following conditions:

(i) for all $t\in(0,1)$, there exists $\psi(t)\in(t,1]$, such that for all $x,y\in P$,
$$A(tx,t^{-1}y)\geq\psi(t)A(x,y).$$

(ii) for all $t\in(0,1)$, $x,y\in P$,
$$B(tx,t^{-1}y)\geq tB(x,y).$$

To our knowledge, the fixed point results for the operator equation (1.1) are still under development. Our results in this paper will extend and improve many known results in the field and in particular those in [2, 7, 21, 27, 28, 32]. The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas to be used to prove our main results. In Section 3, we investigate the existence and uniqueness of positive solutions to the operator equation (1.1) in ordered Banach spaces. In Section 4, to demonstrate the applicability of our abstract results, we give an application to nonlinear fractional differential equation two-point boundary value problems. Finally, we give an example to demonstrate the application of our theoretical results.

2. Preliminaries and Lemmas

In this section, we state some definitions, notations and known results. For convenience of readers, we suggest that one refer to [3, 7, 8, 17, 18] for details.

Suppose that $(E, \|\cdot\|)$ is a Banach space and we denote the zero element of E by θ. Let P be a non-empty closed convex subset of E. We say that P is a cone in E if it satisfies

(1) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$;

(2) $x \in P, -x \in P \Rightarrow x = \theta$.

The Banach space E partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. The cone P is called normal if there exists a constant $N > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and the smallest N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > 0$, we denote by P_h the set $P_h = \{x \in P \mid x \sim h\}$. It is easy to see that $P_h \subset P$. A cone P is said to be solid if its interior $\overset{\circ}{P}$ is non-empty. If P is a solid cone, take any $h \in \overset{\circ}{P}$, then $P_h = \overset{\circ}{P}$.

Definition 2.1. $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x, and decreasing in y, i.e., $u_i, v_i \in P$ $(i = 1, 2)$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.2. An operator $A : P \to P$ is said to be α-concave if there exists $\alpha \in (0, 1)$ such that

$$A(tx) \geq t^{\alpha} Ax, \text{ for all } t \in (0, 1), \ x \in P. \tag{2.1}$$

Definition 2.3. An operator $B : P \to P$ is said to be sub-homogeneous if it satisfies

$$B(tx) \geq tBx, \text{ for all } t \in (0, 1), x \in P. \tag{2.2}$$

Definition 2.4. An operator $C : E \to E$ is said to be homogeneous if it satisfies

$$C(\lambda x) = \lambda Cx, \text{ for all } \lambda > 0, x \in E. \tag{2.3}$$

Remark 2.5. Obviously, a homogeneous operator is a sub-homogeneous operator.

Lemma 2.6 ([32]). *Let P be a normal cone in E. Assume that $T : P \times P \to P$ is a mixed monotone operator and satisfies:*

(A_1) *there exists $h \in P_h$ with $h \neq \theta$ such that $T(h, h) \in P_h$;*

(A_2) *for any $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that*

$$T(tu, t^{-1}v) \geq \varphi(t)T(u, v), \ \textit{for all } u, v \in P.$$

Then

(1) $T : P_h \times P_h \to P_h$;

(2) there exist $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that*

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq T(u_0, v_0) \leq T(v_0, u_0) \leq v_0;$$

(3) T *has a unique fixed point* x^* *in* P_h*;*

(4) for any initial values $x_0, y_0 \in P_h$*, by constructing successively the sequences as follows*

$$x_n = T(x_{n-1}, y_{n-1}), \quad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, ..., \tag{2.4}$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

3. Main results

In this section we consider the existence and uniqueness of a positive solution for the operator equation (1.1). We always assume that E is a real Banach space with a partial order introduced by a normal cone P of E. Take $h \in E,\ h > \theta$, and P_h is given as in Section 2.

Theorem 3.1. *Let* P *be a normal cone in* E*, and let* $A, B : P \times P \to P$ *be two mixed monotone operators and satisfy the following conditions:*

(1) for all $t \in (0,1)$*, there exists* $\psi(t) \in (t, 1]$ *such that*

$$A(tx, t^{-1}y) \geq \psi(t) A(x,y), \text{ for all } x, y \in P; \tag{3.1}$$

(2) for all $t \in (0,1), x, y \in P$,

$$B(tx, t^{-1}y) \geq tB(x,y); \tag{3.2}$$

(3) there exists $h \in P$ *with* $h > \theta$ *such that* $A(h,h) \in P_h$ *and* $B(h,h) \in P_h$*;*

(4) there exists a constant $\delta > 0$*, such that for all* $x, y \in P$,

$$A(x,y) \geq \delta B(x,y). \tag{3.3}$$

Then the operator equation (1.1) *has a unique solution* x^* *in* P_h*, and for any initial values* $x_0, y_0 \in P_h$*, by constructing successively the sequences as follows*

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}),$$
$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}), n = 1, 2, ...,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *in* E *as* $n \to \infty$.

Proof. Firstly, from (3.1) and (3.2), for any $t \in (0,1)$, $x, y \in P$, we have

$$A(t^{-1}x, ty) \leq (\psi(t))^{-1} A(x,y) \tag{3.4}$$

and

$$B(t^{-1}x, ty) \leq t^{-1} B(x,y). \tag{3.5}$$

Since $A(h,h) \in P_h$, $B(h,h) \in P_h$, there exist constants $a_i > 0,\ b_i > 0\ (i = 1,2)$ such that

$$a_1 h \leq A(h,h) \leq b_1 h, \tag{3.6}$$

$$a_2 h \leq B(h,h) \leq b_2 h. \tag{3.7}$$

Next we show $A : P_h \times P_h \to P_h$. For any $x, y \in P_h$, we can choose two sufficiently small numbers $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\alpha_1 h \leq x \leq \frac{1}{\alpha_1} h, \quad \alpha_2 h \leq y \leq \frac{1}{\alpha_2} h. \tag{3.8}$$

Let $\alpha = \min\{\alpha_1, \alpha_2\}$, then $\alpha \in (0, 1)$, by (3.1), (3.4), (3.6) and (3.8), we have

$$A(x, y) \leq A\left(\frac{1}{\alpha}h, \alpha h\right) \leq \frac{1}{\psi(\alpha)}A(h, h) \leq \frac{1}{\psi(\alpha)}b_1 h,$$

$$A(x, y) \geq A\left(\alpha h, \frac{1}{\alpha}h\right) \geq \psi(\alpha)A(h, h) \geq \psi(\alpha)a_1 h.$$

Evidently, $\frac{1}{\psi(\alpha)}b_1,\ \psi(\alpha)a_1 > 0$. Thus $A(x, y) \in P_h$; that is, $A : P_h \times P_h \to P_h$.

Finally, we show $B : P_h \times P_h \to P_h$. For any $x, y \in P_h$, we can choose two sufficiently small numbers $\beta_1, \beta_2 \in (0, 1)$ such that

$$\beta_1 h \leq x \leq \frac{1}{\beta_1}h, \quad \beta_2 h \leq y \leq \frac{1}{\beta_2}h. \tag{3.9}$$

Let $\beta = \min\{\beta_1, \beta_2\}$, then $\beta \in (0, 1)$, by (3.2), (3.5), (3.7) and (3.9), we have

$$B(x, y) \leq B\left(\frac{1}{\beta}h, \beta h\right) \leq \frac{1}{\beta}B(h, h) \leq \frac{1}{\beta}b_2 h,$$

$$B(x, y) \geq B\left(\beta h, \frac{1}{\beta}h\right) \geq \beta B(h, h) \geq \beta a_2 h.$$

Evidently, $\frac{1}{\beta}b_2,\ \beta a_2\ > 0$. Thus $B(x, y) \in P_h$; that is, $B : P_h \times P_h \to P_h$.

Now we define the operator $T = A + B : P \times P \to P$ by

$$T(x, y) = A(x, y) + B(x, y), \quad x, y \in P. \tag{3.10}$$

Then $T : P \times P \to P$ is a mixed monotone operator. Since $A(h, h) \in P_h$, $B(h, h) \in P_h$, we get $T(h, h) = A(h, h) + B(h, h) \in P_h$.

In the following, we show that for any $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that for all $x, y \in P$, $T(tx, t^{-1}y) \geq \varphi(t)T(x, y)$. For any $x, y \in P$, by (3.3), we have

$$A(x, y) + \delta A(x, y) \geq \delta B(x, y) + \delta A(x, y). \tag{3.11}$$

It follows from (3.11) that

$$A(x, y) \geq \frac{A(x, y) + B(x, y)}{1 + \delta^{-1}} = \frac{T(x, y)}{1 + \delta^{-1}}. \tag{3.12}$$

By (3.1), (3.2), (3.10) and (3.12), for all $x, y \in P$, we have

$$\begin{aligned} T(tx, t^{-1}y) - tT(x, y) =& A(tx, t^{-1}y) + B(tx, t^{-1}y) - t(A(x, y) + B(x, y)) \\ \geq& (\psi(t) - t)A(x, y) + B(tx, t^{-1}y) - tB(x, y) \\ \geq& (\psi(t) - t)A(x, y) \\ \geq& \frac{\psi(t) - t}{1 + \delta^{-1}}T(x, y). \end{aligned} \tag{3.13}$$

It follows from (3.13) that for all $x, y \in P$,

$$\begin{aligned} T(tx, t^{-1}y) \geq& tT(x, y) + \frac{\psi(t) - t}{1 + \delta^{-1}}T(x, y) \\ =& \left(t + \frac{\psi(t) - t}{1 + \delta^{-1}}\right)T(x, y). \end{aligned} \tag{3.14}$$

Let $\varphi(t) = t + \frac{\psi(t)-t}{1+\delta^{-1}}$, then $\varphi(t) \in (t, \psi(t)) \subset (t, 1]$, $t \in (0, 1)$ and

$$T(tx, t^{-1}y) \geq \varphi(t)T(x, y), \text{ for all } x, y \in P. \tag{3.15}$$

By Lemma 2.6, the conclusions of Theorem 3.1 holds. □

Remark 3.2. Theorem 3.1 extends the main results in [7, 21, 27, 28]. In fact, in Theorem 3.1 of [21], $A : P \to P$ is an increasing sub-homogeneous operator and $B : P \to P$ is a decreasing operator which satisfies

$$B(t^{-1}y) \geq tBy, \text{ for all } t \in (0,1), x, y \in P.$$

Define a mixed monotone operator $D : P \times P \to P$ by $D(x,y) = Ax + By, x, y, \in P$, then D satisfies

$$D(tx, t^{-1}y) \geq tD(x,y), \text{ for all } t \in (0,1), \ x, y \in P.$$

Therefore, our Theorem 3.1 generalizes and improves Theorem 3.1 of [21].

Remark 3.3. Taking $\psi(t) = t^\alpha$ in Theorem 3.1, we get the following corollary which generalizes and improves Theorem 3.1 of [21].

Corollary 3.4. *Let P be a normal cone in E. Let $A, B : P \times P \to P$ be two mixed monotone operators and satisfy the following conditions:*

(1) there exists $\alpha \in (0,1)$ such that for all $t \in (0,1), x, y \in P$,

$$A(tx, t^{-1}y) \geq t^\alpha A(x,y);$$

(2) for all $t \in (0,1), x, y \in P$,

$$B(tx, t^{-1}y) \geq tB(x,y);$$

(3) there exists $h \in P$ with $h > \theta$ such that $A(h,h) \in P_h$ and $B(h,h) \in P_h$;

(4) there exists a constant $\delta > 0$ such that for all $x, y \in P$,

$$A(x,y) \geq \delta B(x,y).$$

Then the operator equation (1.1) *has a unique solution x^* in P_h, and for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows*

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}), \\ y_n &= A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}), n = 1, 2, ..., \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

If we take $B = \theta$ in Theorem 3.1, then a restructure of the proof (without B) implies an improved version of the main result of [7].

Corollary 3.5. *Let P be a normal cone in E. Let $A : P \times P \to P$ be a mixed monotone operator and satisfies the following conditions:*

(1) for all $t \in (0,1)$, there exists $\psi(t) \in (t,1]$ such that

$$A(tx, t^{-1}y) \geq \psi(t)A(x,y), \text{ for all } x, y \in P;$$

(2) there is $h \in P$ with $h > \theta$ such that $A(h,h) \in P_h$.

Then the operator equation $A(x,x) = x$ has a unique solution x^* in P_h, and for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows

$$x_n = A(x_{n-1}, y_{n-1}), \quad y_n = A(y_{n-1}, x_{n-1}), \ n = 1, 2, ...,$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

Corollary 3.6. *Let P be a normal cone in E, and let $A, B : P \times P \to P$ be two mixed monotone operators and satisfy the following conditions:*

(1) for all $t \in (0,1)$, there exists $\psi(t) \in (t,1]$ such that

$$A(tx, t^{-1}y) \geq \psi(t)A(x,y), \text{ for all } x, y \in P;$$

(2) for all $t \in (0,1), x, y \in P$,

$$B(tx, t^{-1}y) \geq tB(x,y);$$

(3) there is $h \in P$ with $h > \theta$ such that $A(h,h) \in P_h$ and $B(h,h) \in P_h$;

(4) there exists a constant $\delta > 0$ such that for all $x, y \in P$,

$$A(x,y) \geq \delta B(x,y).$$

Then the operator equation $A(x,x) + B(x,x) = \lambda x$ has a unique solution x_λ in P_h for any $\lambda > 0$, and for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows

$$\begin{aligned} x_n &= \frac{1}{\lambda}[A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1})], \\ y_n &= \frac{1}{\lambda}[A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1})], n = 1, 2, ..., \end{aligned}$$

we have $x_n \to x_\lambda$ and $y_n \to x_\lambda$ in E as $n \to \infty$.

Proof. It is obvious that the operator $\lambda^{-1}(A+B)(\lambda > 0)$ in Corollary 3.5 satisfies the conditions of Theorem 3.1, thus it follows from Theorem 3.1 that the conclusion of Corollary 3.5 holds. □

Corollary 3.7. *Let P be a normal cone in E. Let $h > \theta$, and $A, B : P_h \times P_h \to P_h$ be two mixed monotone operators and satisfy the following conditions:*

(1) there exists $\alpha \in (0,1)$ such that for all $t \in (0,1), x, y \in P_h$,

$$A(tx, t^{-1}y) \geq t^\alpha A(x,y);$$

(2) for all $t \in (0,1), x, y \in P_h$,

$$B(tx, t^{-1}y) \geq tB(x,y);$$

(3) there exists a constant $\delta > 0$ such that for all $x, y \in P_h$,

$$A(x,y) \geq \delta B(x,y).$$

Then the operator equation $A(x,x) + B(x,x) = x$ has a unique solution x^* in P_h, and for any initial values $x_0, y_0 \in P_h$, by constructing successively the sequences as follows

$$\begin{aligned} x_n &= A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}), \\ y_n &= A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}), n = 1, 2, ..., \end{aligned}$$

we have $x_n \to x^*$ and $y_n \to x^*$ in E as $n \to \infty$.

Remark 3.8. If P is a solid cone, take $h \in \mathring{P}$, then $P_h = \mathring{P}$. If $A, B : P_h \times P_h \to P_h$ or $\mathring{P} \times \mathring{P} \to \mathring{P}$, then $A(h,h) \in P_h$ and $B(h,h) \in P_h$ are automatically satisfied in Corollary 3.6.

Theorem 3.9. *Suppose that all the conditions of* Theorem 3.1 *are satisfied. Then the operator equation*

$$A(x,x)+B(x,x)=\lambda x,\ \ \lambda>0 \tag{3.16}$$

has a unique solution x_λ which satisfies

(1) if there exists $\beta\in(0,1)$ such that $\psi(t)\geq\frac{t^\beta-t}{\delta}+t^\beta$ for $t\in(0,1)$, then x_λ is continuous in $\lambda\in(0,\infty)$, that is $\lambda\to\lambda_0$ $(\lambda_0>0)$ implies $\|x_\lambda-x_{\lambda_0}\|\to 0$;

(2) if $\psi(t)>\frac{t^{\frac{1}{2}}-t}{\delta}+t^{\frac{1}{2}}$ for $t\in(0,1)$, then x_λ is strictly decreasing in λ, that is, $0<\lambda_1<\lambda_2$ implies $x_{\lambda_1}>x_{\lambda_2}$;

(3) if there exists $\beta\in(0,\frac{1}{2})$ such that $\psi(t)\geq\frac{t^\beta-t}{\delta}+t^\beta$ for $t\in(0,1)$, then $\lim\limits_{\lambda\to\infty}\|x_\lambda\|=0$, $\lim\limits_{\lambda\to 0^+}\|x_\lambda\|=\infty$.

Proof. It is obvious that the operator $\lambda^{-1}(A+B)(\lambda>0)$ satisfies the conditions of Theorem 3.1, thus it follows from Theorem 3.1 that the operator equation (3.16) has a unique solution x_λ in P. For convenience of proof, we let

$$\alpha(t)=\frac{\ln(t+\delta\psi(t))-\ln(1+\delta)}{\ln t},\ \text{ for all } t\in(0,1).$$

Then $\alpha(t)\in(0,1)$ and $t^{\alpha(t)}=t+\frac{\psi(t)-t}{1+\delta^{-1}}$.

(1) For any given $0<\lambda_1<\lambda_2$, by Theorem 3.1, the operator equation (3.16) has unique solutions $x_{\lambda_1},x_{\lambda_2}\in P_h$ respectively, so $x_{\lambda_1}\sim x_{\lambda_2}$. Thus there exists a positive number $d>0$, such that $\frac{1}{d}x_{\lambda_2}\leq x_{\lambda_1}\leq dx_{\lambda_2}$. So, $\{t>0\mid x_{\lambda_1}\geq tx_{\lambda_2},x_{\lambda_2}\geq tx_{\lambda_1}\}\neq\emptyset$. Let $t_0=\sup\{t>0\mid x_{\lambda_1}\geq tx_{\lambda_2},x_{\lambda_2}\geq tx_{\lambda_1}\}$. It is obvious that $0<t_0\leq 1$ and

$$x_{\lambda_1}\geq t_0x_{\lambda_2},\quad x_{\lambda_2}\geq t_0x_{\lambda_1}. \tag{3.17}$$

If $t_0=1$, then by (3.17), $x_{\lambda_1}=x_{\lambda_2}$. By Theorem 3.1, for fixed $\lambda>0$, the operator equation (3.16) has a unique solution. which is a contradiction with $\lambda_1<\lambda_2$, thus $0<t_0<1$. Note that $t^{\alpha(t)}=t+\frac{\psi(t)-t}{1+\delta^{-1}}$, thus we have

$$\begin{aligned}
\lambda_1x_{\lambda_1}&=A\left(x_{\lambda_1},x_{\lambda_1}\right)+B\left(x_{\lambda_1},x_{\lambda_1}\right)\\
&\geq A\left(t_0x_{\lambda_2},t_0^{-1}x_{\lambda_2}\right)+B\left(t_0x_{\lambda_2},t_0^{-1}x_{\lambda_2}\right)\\
&\geq\psi(t_0)A(x_{\lambda_2},x_{\lambda_2})+t_0B(x_{\lambda_2},x_{\lambda_2})\\
&\geq\left[\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)+\left[\psi(t_0)-\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)\right]\right]A(x_{\lambda_2},x_{\lambda_2})+t_0B(x_{\lambda_2},x_{\lambda_2})\\
&\geq\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)A(x_{\lambda_2},x_{\lambda_2})+\left[\psi(t_0)-\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)\right]A(x_{\lambda_2},x_{\lambda_2})+t_0B(x_{\lambda_2},x_{\lambda_2})\\
&\geq t_0^{\alpha(t_0)}A(x_{\lambda_2},x_{\lambda_2})+\frac{\psi(t_0)-t_0}{1+\delta}A(x_{\lambda_2},x_{\lambda_2})+t_0B(x_{\lambda_2},x_{\lambda_2})\\
&\geq t_0^{\alpha(t_0)}A(x_{\lambda_2},x_{\lambda_2})+\frac{\delta\left(\psi(t_0)-t_0\right)}{1+\delta}B(x_{\lambda_2},x_{\lambda_2})+t_0B(x_{\lambda_2},x_{\lambda_2})\\
&=t_0^{\alpha(t_0)}A(x_{\lambda_2},x_{\lambda_2})+\left(\frac{\delta\left(\psi(t_0)-t_0\right)}{1+\delta}+t_0\right)B(x_{\lambda_2},x_{\lambda_2})\\
&=t_0^{\alpha(t_0)}[A(x_{\lambda_2},x_{\lambda_2})+B(x_{\lambda_2},x_{\lambda_2})]\\
&=t_0^{\alpha(t_0)}\lambda_2x_{\lambda_2},
\end{aligned}$$

where $\alpha(t_0)=\frac{\ln(t_0+\delta\psi(t_0))-\ln(1+\delta)}{\ln t_0}\in(0,1)$. In a similar way, we have

$$\begin{aligned}
\lambda_2 x_{\lambda_2} &= A\left(x_{\lambda_2},x_{\lambda_2}\right)+B\left(x_{\lambda_2},x_{\lambda_2}\right)\\
&\geq A\left(t_0x_{\lambda_1},t_0^{-1}x_{\lambda_1}\right)+B\left(t_0x_{\lambda_1},t_0^{-1}x_{\lambda_1}\right)\\
&\geq \psi(t_0)A(x_{\lambda_1},x_{\lambda_1})+t_0B(x_{\lambda_1},x_{\lambda_1})\\
&\geq \left[\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)+\left[\psi(t_0)-\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)\right]\right]A(x_{\lambda_1},x_{\lambda_1})+t_0B(x_{\lambda_1},x_{\lambda_1})\\
&\geq \left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)A(x_{\lambda_1},x_{\lambda_1})+\left[\psi(t_0)-\left(t_0+\frac{\psi(t_0)-t_0}{1+\delta^{-1}}\right)\right]A(x_{\lambda_1},x_{\lambda_1})+t_0B(x_{\lambda_1},x_{\lambda_1})\\
&\geq t_0^{\alpha(t_0)}A(x_{\lambda_1},x_{\lambda_1})+\frac{\psi(t_0)-t_0}{1+\delta}A(x_{\lambda_1},x_{\lambda_1})+t_0B(x_{\lambda_1},x_{\lambda_1})\\
&\geq t_0^{\alpha(t_0)}A(x_{\lambda_1},x_{\lambda_1})+\frac{\delta\left(\psi(t_0)-t_0\right)}{1+\delta}B(x_{\lambda_1},x_{\lambda_1})+t_0B(x_{\lambda_1},x_{\lambda_1})\\
&= t_0^{\alpha(t_0)}A(x_{\lambda_1},x_{\lambda_1})+\left(\frac{\delta\left(\psi(t_0)-t_0\right)}{1+\delta}+t_0\right)B(x_{\lambda_1},x_{\lambda_1})\\
&= t_0^{\alpha(t_0)}[A(x_{\lambda_1},x_{\lambda_1})+B(x_{\lambda_1},x_{\lambda_1})]\\
&= t_0^{\alpha(t_0)}\lambda_1x_{\lambda_1}.
\end{aligned}$$

Thus,

$$x_{\lambda_1}\geq\lambda_1^{-1}\lambda_2t_0^{\alpha(t_0)}x_{\lambda_2},\quad x_{\lambda_2}\geq\lambda_2^{-1}\lambda_1t_0^{\alpha(t_0)}x_{\lambda_1}. \tag{3.18}$$

Noting that $\lambda_1^{-1}\lambda_2t_0^{\alpha(t_0)}>\lambda_1^{-1}\lambda_2t_0>t_0$, by (3.18) and the definition of t_0, we have $\lambda_2^{-1}\lambda_1t_0^{\alpha(t_0)}\leq t_0$, that is

$$t_0\geq\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}}. \tag{3.19}$$

Since $\psi(t)\geq\frac{t^\beta-t}{\delta}+t^\beta$ for all $t\in(0,1)$, we have $t^{\alpha(t)}=t+\frac{\psi(t)-t}{1+\delta^{-1}}\geq t^\beta$ for all $t\in(0,1)$. Thus $\alpha(t)=\frac{\ln(t+\delta\psi(t))-\ln(1+\delta)}{\ln t}\leq\beta$ for all $t\in(0,1)$. From (3.18) and (3.19), we have

$$\begin{aligned}
&\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}x_{\lambda_2}\leq\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}}x_{\lambda_2}\leq x_{\lambda_1}\leq\left(\tfrac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0)}}x_{\lambda_2}\leq\left(\tfrac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}}x_{\lambda_2},\\
&\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}x_{\lambda_1}\leq\left(\tfrac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}}x_{\lambda_1}\leq x_{\lambda_2}\leq\left(\tfrac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0)}}x_{\lambda_1}\leq\left(\tfrac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}}x_{\lambda_1}.
\end{aligned} \tag{3.20}$$

By (3.20) and the normality of P, we have

$$\|x_{\lambda_1}-x_{\lambda_2}\|\to0,\quad\lambda_1\to\lambda_2^-,$$

$$\|x_{\lambda_2}-x_{\lambda_1}\|\to0,\quad\lambda_2\to\lambda_1^+.$$

So the conclusion (1) holds.

(2) By (3.18) and (3.19), we get

$$x_{\lambda_1}\geq\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}}x_{\lambda_2}. \tag{3.21}$$

Noting that $\psi(t_0) > \frac{t_0^{\frac{1}{2}}-t_0}{\delta} + t_0^{\frac{1}{2}}$, we get $\alpha(t_0) \in (0, \frac{1}{2})$, and consequently we have

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} > 1,$$

and by (3.21) the conclusion (2) holds.

(3) Since $\psi(t) \geq \frac{t^\beta - t}{\delta} + t^\beta$ for $t \in (0,1)$, we have $t^{\alpha(t)} = t + \frac{\psi(t)-t}{1+\delta^{-1}} \geq t^\beta$ for $t \in (0,1)$, thus we have $\alpha(t) \leq \beta < \frac{1}{2}$ for $t \in (0,1)$. Let $\lambda_1 = 1,\ \lambda_2 = \lambda$ in (3.21), then we have

$$x_1 \geq \lambda^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} x_\lambda \geq \lambda^{\frac{1-2\beta}{1-\beta}} x_\lambda, \quad \text{for all } \lambda > 1.$$

Thus

$$\|x_\lambda\| \leq N\lambda^{-\frac{1-2\beta}{1-\beta}} \|x_1\|, \quad \text{for all } \lambda > 1,$$

where N is the normality constant. Let $\lambda \to \infty$, then $\|x_\lambda\| \to 0$. Let $\lambda_1 = \lambda,\ \lambda_2 = 1$ in (3.21), we have

$$x_\lambda \geq \lambda^{-\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} x_1 \geq \lambda^{-\frac{1-2\beta}{1-\beta}} x_1, \quad \text{for all } 0 < \lambda < 1.$$

Thus

$$\|x_\lambda\| \geq N^{-1}\lambda^{-\frac{1-2\beta}{1-\beta}} \|x_1\|, \quad \text{for all } 0 < \lambda < 1,$$

where N is the normality constant. Let $\lambda \to 0^+$, then $\|x_\lambda\| \to \infty$. So the conclusion (3) holds. □

Remark 3.10. If $\delta > 3$, then $t < \frac{t^{\frac{1}{2}}-t}{\delta} + t^{\frac{1}{2}} < \frac{t^{\frac{1}{3}}-t}{\delta} + t^{\frac{1}{3}} < \frac{t^{\frac{1}{4}}-t}{\delta} + t^{\frac{1}{4}} < 1$, for all $t \in (0,1)$. Taking $\psi(t) = \frac{t^{\frac{1}{4}}-t}{\delta} + t^{\frac{1}{4}},\ t \in (0,1)$, we get that ψ satisfies the conditions of Theorem 3.9.

4. Application

In this section, we apply the results in Section 3 to study nonlinear fractional differential equations with two-point boundary conditions. We here consider the existence and uniqueness of positive solutions for the following fractional boundary value problem (FBVP for short):

$$\begin{cases} -D_{0+}^{\alpha} u(t) = F(t, u(t)) + G(t, u(t)),\ 0 < t < 1,\ n-1 < \alpha \leq n, \\ u^i(0) = 0,\ 0 \leq i \leq n-2, \\ [D_{0+}^{\beta} u(t)]_{t=1} = 0, \quad 1 \leq \beta \leq n-2, \end{cases} \tag{4.1}$$

where $D_{0+}^{\alpha} u(t)$ is the Riemann-Liouville fractional derivative of order $\alpha, n > 2, n \in \mathbb{N}$.

Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the wide range of applications of such kind of equations in various scientific fields such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc., see [18, 20, 21, 28]. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and many results have been achieved, and here we refer the reader to [5, 6, 12, 14, 15, 19, 25, 29, 30, 31, 32, 35, 36, 37] and the references therein for details.

However, not much work has been done to utilize the fixed point results on mixed monotone operators with perturbation to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems. This motivates us to investigate FBVP (4.1) by using our new fixed point theorems presented in Section 3. It will be shown that our results not only can guarantee the existence of a unique positive solution, but also can be applied to construct an iterative scheme for approximating the solution.

Definition 4.1 ([18]). Let $\alpha > 0$ with $\alpha \in \mathbb{R}$. Suppose that $u : (0, +\infty) \to \mathbb{R}$. Then the Riemann-Liouville fractional derivative of order α is defined as

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α.

Lemma 4.2 ([6]). *Let $u \in C[0,1]$, then the fractional boundary value problem*

$$\begin{cases} -D_{0+}^{\alpha} u(t) = g(t), \ 0 < t < 1, \ n-1 < \alpha \le n, \\ u^i(0) = 0, \ 0 \le i \le n-2, \\ [D_{0+}^{\beta} u(t)]_{t=1} = 0, \quad 1 \le \beta \le n-2 \end{cases}$$

has a unique positive solution

$$u(t) = \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, \ 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \ 0 \le t \le s \le 1. \end{cases}$$

Lemma 4.3 ([6, 29]). *The Green function $G(t,s)$ in* Lemma 4.2 *has the following properties:*

(1) $G(t,s)$ is continuous on $[0,1] \times [0,1]$;

(2) for all $(t,s) \in [0,1] \times [0,1]$, we have $G(t,s) \ge 0$;

(3) for all $t, s \in [0,1]$, we have

$$[1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1} t^{\alpha-1} \le \Gamma(\alpha) G(t,s) \le (1-s)^{\alpha-\beta-1} t^{\alpha-1}.$$

Let $E = C[0,1]$, $\|u\| = \sup\{u(t) \mid t \in [0,1]\}$, $P = \{u \in C[0,1] \mid u(t) \ge 0, t \in [0,1]\}$. It is clear that E is a Banach space and P is a normal cone of E.

Theorem 4.4. *Assume that $F(t,x) = f(t,x,x), G(t,x) = g(t,x,x)$ and satisfying the following conditions (H_1)– (H_4):*

(H_1) *$f, \ g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous, and for all $t \in [0,1]$, $g(t,0,1) \not\equiv 0$;*

(H_2) *for fixed $t \in [0,1]$ and $y \in [0,+\infty)$, $f(t,x,y), \ g(t,x,y)$ are increasing in $x \in [0,+\infty)$; for fixed $t \in [0,1]$ and $x \in [0,+\infty)$, $f(t,x,y), \ g(t,x,y)$ are decreasing in $y \in [0,+\infty)$;*

(H_3) *for all $\lambda \in (0,1)$, there exists $\psi(\lambda) \in (\lambda,1)$ such that for all $t \in [0,1], \ x,y \in [0,+\infty)$, $f(t,\lambda x,\lambda^{-1}y) \ge \psi(\lambda) f(t,x,y)$, $g(t,\lambda x,\lambda^{-1}y) \ge \lambda g(t,x,y)$;*

(H_4) *there exists a constant $\delta > 0$, such that for all $t \in [0,1], \ x,y \in [0,+\infty)$, $f(t,x,y) \ge \delta g(t,x,y)$.*

Then the problem (4.1) has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,1]$, and for any $u_0, v_0 \in P_h$, by constructing successively the sequences as follows

$$u_{n+1}(t) = \int_0^1 G(t,s)[f(s,u_n(s),v_n(s)) + g(s,u_n(s),v_n(s))] ds, \ n = 0,1,2,\ldots,$$

$$v_{n+1}(t) = \int_0^1 G(t,s)[f(s,v_n(s),u_n(s)) + g(s,v_n(s),u_n(s))] ds, \ n = 0,1,2,\ldots,$$

we have $u_n(t) \rightrightarrows u^*(t), t \in [0,1]$ and $v_n(t) \rightrightarrows u^*(t), t \in [0,1]$, that is, $\{u_n(t)\}$ and $\{v_n(t)\}$ both converges to $u^*(t)$ uniformly for all $t \in [0,1]$.

Proof. From [25], the problem (4.1) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s)\left[f\left(s,u(s),u(s)\right) + g\left(s,u(s),u(s)\right)\right] ds,$$

where $G(t,s)$ is as given in Lemma 4.3.

Define two operators $A, B : P \times P \to E$ by

$$A(u,v)(t) = \int_0^1 G(t,s) f\left(s,u(s),v(s)\right) ds,$$

$$B(u,v)(t) = \int_0^1 G(t,s) g\left(s,u(s),v(s)\right) ds.$$

It is easy to prove that u is the solution of the problem (4.1) if and only if $u = A(u,u) + B(u,u)$. From (H_1), we know that $A, B : P \times P \to P$.

(1) Firstly, we prove that $A,\ B$ are two mixed monotone operators. In fact, for all $u_i, v_i \in P$ $(i = 1,2)$ with $u_1 \geq u_2,\ v_1 \leq v_2$, by (H_2) we get that $u_1(t) \geq u_2(t), v_1(t) \leq v_2(t), t \in [0,1]$ and it follows from (H_2) that

$$\begin{aligned} A(u_1,v_1)(t) &= \int_0^1 G(t,s) f\left(s,u_1(s),v_1(s)\right) ds \\ &\geq \int_0^1 G(t,s) f\left(s,u_2(s),v_2(s)\right) ds \\ &= A(u_2,v_2)(t). \end{aligned}$$

That is $A(u_1,v_1) \geq A(u_2,v_2)$. In a similar way we get $B(u_1,v_1) \geq B(u_2,v_2)$.

(2) From (H_3), for any $\lambda \in (0,1)$, $t \in [0,1]$ and $u, v \in P$, we have

$$\begin{aligned} A(\lambda u, \lambda^{-1} v)(t) &= \int_0^1 G(t,s) f\left(s, \lambda u(s), \lambda^{-1} v(s)\right) ds \\ &\geq \psi(\lambda) \int_0^1 G(t,s) f\left(s,u(s),v(s)\right) ds \\ &= \psi(\lambda) A(u,v)(t), \\ B(\lambda u, \lambda^{-1} v)(t) &= \int_0^1 G(t,s) g\left(s, \lambda u(s), \lambda^{-1} v(s)\right) ds \\ &\geq \lambda \int_0^1 G(t,s) g\left(s,u(s),v(s)\right) ds \\ &= \lambda B(u,v)(t). \end{aligned}$$

That is, for any $\lambda \in (0,1),\ u, v \in P,\ A(\lambda u, \lambda^{-1} v) \geq \psi(\lambda) A(u,v),\ B(\lambda u, \lambda^{-1} v) \geq \lambda B(u,v)$.

(3) Next we show that $A(h,h) \in P_h,\ B(h,h) \in P_h$. In fact, from Lemma 4.3 and (H_2), we have

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s) f(s,h(s),h(s)) ds \\ &\leq \int_0^1 G(t,s) f(s,1,0) ds \\ &\leq \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-\beta-1} f(s,1,0) ds. \end{aligned}$$

On the other hand, by the property (3) of Green function in Lemma 4.3 and (H_2), we have

$$\begin{aligned} A(h,h)(t) &= \int_0^1 G(t,s)f(s,h(s),h(s))ds \\ &\geq \int_0^1 G(t,s)f(s,0,1)ds \\ &\geq \frac{1}{\Gamma(\alpha)}h(t)\int_0^1 [1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1}f(s,0,1)ds. \end{aligned}$$

By (H_2) and (H_4), we have

$$f(s,1,0) \geq f(s,0,1) \geq \delta g(s,0,1).$$

It follows from $g(t,0,1) \not\equiv 0$ for all $t \in [0,1]$ that

$$\int_0^1 f(s,1,0)ds \geq \int_0^1 f(s,0,1)ds \geq \int_0^1 \delta g(s,0,1)ds > 0.$$

Let

$$\begin{aligned} l_1 &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}f(s,1,0)ds > 0, \\ l_2 &= \frac{1}{\Gamma(\alpha)}\int_0^1 [1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1}f(s,0,1)ds > 0. \end{aligned}$$

Thus, $l_2h(t) \leq A(h,h)(t) \leq l_1h(t),\ t \in [0,1]$, and we have $A(h,h) \in P_h$. In a similar way, we get

$$\begin{aligned} B(h,h)(t) &= \int_0^1 G(t,s)g(s,h(s),h(s))ds \\ &\leq \int_0^1 G(t,s)g(s,1,0)ds \\ &\leq \frac{1}{\Gamma(\alpha)}h(t)\int_0^1 (1-s)^{\alpha-\beta-1}g(s,1,0)ds, \end{aligned}$$

$$\begin{aligned} B(h,h)(t) &= \int_0^1 G(t,s)g(s,h(s),h(s))ds \\ &\geq \int_0^1 G(t,s)g(s,0,1)ds \\ &\geq \frac{1}{\Gamma(\alpha)}h(t)\int_0^1 [1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1}g(s,0,1)ds. \end{aligned}$$

Let

$$\begin{aligned} l_3 &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1}g(s,1,0)ds > 0, \\ l_4 &= \frac{1}{\Gamma(\alpha)}\int_0^1 [1-(1-s)^{\beta}](1-s)^{\alpha-\beta-1}g(s,0,1)ds > 0. \end{aligned}$$

Thus, $l_4h(t) \leq B(h,h)(t) \leq l_3h(t),\ t \in [0,1]$, and we have $B(h,h) \in P_h$.

(4) For any $u, v \in P$, $t \in [0,1]$, from (H_4) we know that

$$\begin{aligned} A(u,v)(t) &= \int_0^1 G(t,s)f(s,u(s),v(s))ds \\ &\geq \delta\int_0^1 G(t,s)g(s,u(s),v(s))ds \\ &= \delta B(u,v)(t). \end{aligned}$$

So we get $A(u,v) \geq \delta B(u,v)$. Hence all the conditions of Theorem 3.1 are satisfied, and the conclusion of Theorem 4.4 holds. □

Example 4.5. Consider the following two-point boundary value problem

$$\begin{cases} -D_{0+}^{\alpha}u(t) = 2t + \frac{\sqrt{u+1}}{\sqrt[3]{u+1}} + \frac{\sqrt{u+1}}{\sqrt{u+1}},\ 0<t<1,\ n-1<\alpha\leq n, \\ u^{i}(0)=0,\ 0\leq i\leq n-2, \\ [D_{0+}^{\beta}u(t)]_{t=1}=0, \quad 1\leq\beta\leq n-2. \end{cases} \tag{4.2}$$

The above equations can be written in the form of (4.1) with the functions $f,\ g:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ defined by

$$f(t,x,y) = t + \frac{\sqrt{x+1}}{\sqrt[3]{y+1}}, \quad t\in[0,1],\ x,y\geq 0,$$

$$g(t,x,y) = t + \frac{\sqrt{x+1}}{\sqrt{y+1}}, \quad t\in[0,1],\ x,y\geq 0.$$

Now we show in the following that all the conditions of Theorem 4.4 are satisfied.

(1) Clearly, the functions $f,g:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are continuous with $g(t,0,1)\not\equiv 0$.

(2) We observe that for fixed $t\in[0,1]$ and $y\in[0,+\infty)$, $f(t,x,y),\ g(t,x,y)$ are increasing in $x\in[0,+\infty)$; for fixed $t\in[0,1]$ and $x\in[0,+\infty)$, $f(t,x,y),\ g(t,x,y)$ are decreasing in $y\in[0,+\infty)$.

(3) For all $\lambda\in(0,1),\ t\in[0,1]$ and $x\geq 0, y\geq 0$, taking $\psi(\lambda)=\lambda^{\frac{5}{6}}\in(0,1)$, we have

$$\begin{aligned} f(t,\lambda x,\lambda^{-1}y) &= t + \frac{\sqrt{\lambda x+1}}{\sqrt[3]{\lambda^{-1}y+1}} \geq t + \frac{\sqrt{\lambda x+\lambda}}{\sqrt[3]{\lambda^{-1}y+\lambda^{-1}}} \\ &= t + \frac{\lambda^{\frac{1}{2}}\sqrt{x+1}}{\lambda^{-\frac{1}{3}}\sqrt[3]{y+1}} \geq \lambda^{\frac{5}{6}}t + \lambda^{\frac{5}{6}}\frac{\sqrt{x+1}}{\sqrt[3]{y+1}} \\ &= \psi(\lambda)f(t,x,y). \end{aligned}$$

For all $\lambda\in(0,1),\ t\in[0,1]$ and $x\geq 0, y\geq 0$, we have

$$\begin{aligned} g(t,\lambda x,\lambda^{-1}y) &= t + \frac{\sqrt{\lambda x+1}}{\sqrt{\lambda^{-1}y+1}} \geq t + \frac{\sqrt{\lambda x+\lambda}}{\sqrt{\lambda^{-1}y+\lambda^{-1}}} \\ &= t + \frac{\lambda^{\frac{1}{2}}\sqrt{x+1}}{\lambda^{-\frac{1}{2}}\sqrt{y+1}} \geq \lambda t + \lambda\frac{\sqrt{x+1}}{\sqrt{y+1}} \\ &= \lambda g(t,x,y). \end{aligned}$$

(4) Taking $\delta=1$, for all $t\in[0,1]$ and $x\geq 0, y\geq 0$, we have

$$f(t,x,y) = t + \frac{\sqrt{x+1}}{\sqrt[3]{y+1}} \geq t + \frac{\sqrt{x+1}}{\sqrt{y+1}} = g(t,x,y).$$

Thus we have proved that all the conditions of Theorem 4.4 are satisfied. Hence we deduce that (4.2) has one and only one positive solution $x^{*}\in P_h$, where $h(t)=t^{\alpha-1},\ t\in[0,1]$.

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