Some fixed point results for nonlinear mappings in convex metric spaces

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Abstract

In this paper, we consider an iteration process to approximate a common random fixed point of a finite family of asymptotically quasi-nonexpansive random mappings in convex metric spaces. Our results extend and improve several known results in recent literature.

Keywords: Asymptotically quasi-nonexpansive random mappings, random iteration process, common random fixed point, convex metric spaces.

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1. Introduction and Preliminaries

Random fixed point theorems are stochastic generalizations of classical fixed point theorems, which are usually used to obtain the solutions of nonlinear random systems [3]. Some random fixed point theorems for random mappings on separable metric spaces were first proved by Spacek [18] and Hans [7]. Itoh [8] introduced multivalued random contractive mappings on separable metric spaces and considered some random fixed point theorems for the mappings. Choudhury [5] gave a random Ishikawa iteration process to converge to fixed points of the given random mappings. After that, many authors [1, 2, 5, 11, 12, 13, 14, 17, 16] have worked on random iterative algorithms for contractive and asymptotically nonexpansive random mappings in separable normed spaces, Banach spaces and uniformly convex Banach spaces.

In 1970, Takahashi [19] introduced a notion of convex metric space which is a more general space, and each linear normed space is a special example of a convex metric space. Recently [4, 10, 21, 22] have discussed different iteration processes to obtain fixed point of asymptotically quasi-nonexpansive mappings in convex metric spaces.

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Inspired and motivated by the above facts, we will construct an iteration process which converges strongly to a common random fixed point of a finite family of asymptotically quasi-nonexpansive random mappings in convex metric spaces. The results extend and improve the corresponding results in [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [20] [21] [22].

Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ being a $\sigma$-algebra of subsets of $\Omega$, and let $K$ be a nonempty subset of a metric space $(X, d)$.  

**Definition 1.1** ([1]). (i) A mapping $\xi : \Omega \to X$ is measurable if $\xi^{-1}(U) \in \Sigma$ for each open subset $U$ of $X$;  
(ii) The mapping $T : \Omega \times K \to X$ is a random mapping if and only if for each fixed $x \in K$, the mapping $T(\cdot, x) : \Omega \to X$ is measurable, and it is continuous if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot) : K \to X$ is continuous;  
(iii) A measurable mapping $\xi : \Omega \to K$ is a random fixed point of the random mapping $T : \Omega \times K \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

We denote by $\mathbb{N}$ the set of natural numbers, $F(T)$ the set of all random fixed points of a random map $T$ and $T^n(\omega, x)$ the $n$th iteration $T(\omega, T(\omega, T(\omega, \cdots T(\omega, x) \cdots )))$ of $T$ for each $\omega \in \Omega$. The letter $I$ denotes the random mapping $T : \Omega \times K \to K$ defined by $I(\omega, x) = x$ and $T^0 = I$ for each $\omega \in \Omega$.

Next, we introduce some random mappings in metric spaces.

**Definition 1.2.** Let $K$ be a nonempty subset of a separable metric space $(X, d)$ and $T : \Omega \times K \to K$ be a random mapping. The mapping $T$ is said to be  
(i) a nonexpansive random mapping if  
$$d(T(\omega, x), T(\omega, y)) \leq d(x, y)$$  
for each $\omega \in \Omega$ and $x, y \in K$;  
(ii) an asymptotically nonexpansive random mapping if there exists a sequence of measurable mappings \{r_n(\omega)\} : $\Omega \to [0, \infty)$ with $\lim_{n \to \infty} r_n(\omega) = 0$ such that  
$$d(T^n(\omega, x), T^n(\omega, y)) \leq (1 + r_n(\omega))d(x, y)$$  
for each $\omega \in \Omega$, $n \in \mathbb{N}$ and $x, y \in K$;  
(iii) an asymptotically quasi-nonexpansive random mapping if there exists a sequence of measurable mappings \{r_n(\omega)\} : $\Omega \to [0, \infty)$ with $\lim_{n \to \infty} r_n(\omega) = 0$ such that  
$$d(T^n(\omega, \eta(\omega)), \xi(\omega)) \leq (1 + r_n(\omega))d(\eta(\omega), \xi(\omega))$$  
for each $\omega \in \Omega$ and $n \in \mathbb{N}$, where $\xi \in F(T) \neq \emptyset$ and $\eta : \Omega \to K$ is any measurable mapping.  
(iv) a semicompact random mapping if for any sequence of measurable mappings \{\xi_n(\omega)\} : $\Omega \to K$, with $\lim_{n \to \infty} d(T(\omega, \xi_n(\omega)), \xi_n(\omega)) = 0$ for each $\omega \in \Omega$ and $n \in \mathbb{N}$, there exists a subsequence \{\xi_{n_j}\} of \{\xi_n\} which converges pointwise to $\xi$, where $\xi : \Omega \to K$ is a measurable mapping.

**Remark 1.3.** It is easy to see that if $T$ is an asymptotically nonexpansive random mapping and $F(T) \neq \emptyset$, then $T$ is an asymptotically quasi-nonexpansive random mapping.

**Definition 1.4** ([19]). A convex structure in a metric space $(X, d)$ is a mapping $W : X \times X \times [0, 1] \to X$ satisfying, for each $x, y, u \in X$ and each $\lambda \in [0, 1]$  
$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$  
A metric space together with a convex structure is called a convex metric space.

A nonempty subset $K$ of $X$ is said to be convex if $W(x, y; \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times [0, 1]$. The mapping $W : K \times K \times [0, 1] \to K$ is said to be a measurable convex structure if for any two measurable mappings $\xi, \eta : \Omega \to K$ and each fixed $\lambda \in [0, 1]$, the mapping $W(\xi(\cdot), \eta(\cdot); \lambda) : \Omega \to K$ is measurable.
In Banach spaces, Khan et al. [9] introduced the following iteration process for common fixed points of asymptotically quasi-nonexpansive mappings \( \{T_i : i \in J = \{1, 2, \cdots, k\}\} \): any initial point \( x_1 \in K \),

\[
\begin{align*}
    x_{n+1} &= (1 - \alpha_{kn})x_n + \alpha_{kn}T^n_k y_{(k-1)n}, \\
    y_{(k-1)n} &= (1 - \alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T^n_{(k-1)} y_{(k-2)n}, \\
    y_{(k-2)n} &= (1 - \alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T^n_{(k-2)} y_{(k-3)n}, \\
    & \vdots \\
    y_{1n} &= (1 - \alpha_{1n})x_n + \alpha_{1n}T^n_1 y_0, 
\end{align*}
\]

(1.1)

where \( y_0 = x_n \) and \( \{\alpha_{in}\} \) are real sequences in \([0, 1]\) for all \( n \in \mathbb{N} \). And then, Khan and Ahmed [10] considered the iteration process (1.1) in convex metric spaces as follows:

\[
\begin{align*}
    x_{n+1} &= W(T^n_k y_{(k-1)n}, x_n; \alpha_{kn}), \\
    y_{(k-1)n} &= W(T^n_{k-1} y_{(k-2)n}, x_n; \alpha_{(k-1)n}), \\
    y_{(k-2)n} &= W(T^n_{k-2} y_{(k-3)n}, x_n; \alpha_{(k-2)n}), \\
    & \vdots \\
    y_{1n} &= W(T^n_1 y_0, x_n; \alpha_{1n}), 
\end{align*}
\]

(1.2)

where \( y_0 = x_n \) and \( \{\alpha_{in}\} \) are real sequences in \([0, 1]\) for all \( n \in \mathbb{N} \).

From (1.1) and (1.2), we investigate the following random iteration process in convex metric space.

**Definition 1.5.** Let \( \{T_i : i \in J\} \) be a finite family of asymptotically quasi-nonexpansive random mappings from \( \Omega \times K \) to \( K \), where \( K \) is a nonempty closed convex subset of a separable convex metric space \((X, d)\). Let \( \xi_1 : \Omega \to K \) be a measurable mapping, for each \( \omega \in \Omega \), the sequence \( \{\xi_n(\omega)\} \) is defined as follows:

\[
\begin{align*}
    \xi_{n+1}(\omega) &= W(T^n_k (\omega, \eta_{(k-1)n}(\omega)), \xi_n(\omega); \alpha_{kn}), \\
    \eta_{(k-1)n}(\omega) &= W(T^n_{k-1} (\omega, \eta_{(k-2)n}(\omega)), \xi_n(\omega); \alpha_{(k-1)n}), \\
    \eta_{(k-2)n}(\omega) &= W(T^n_{k-2} (\omega, \eta_{(k-3)n}(\omega)), \xi_n(\omega); \alpha_{(k-2)n}), \\
    & \vdots \\
    \eta_{1n}(\omega) &= W(T^n_1 (\omega, \eta_0(\omega)), \xi_n(\omega); \alpha_{1n}), 
\end{align*}
\]

(1.3)

where \( \eta_0(\omega) = \xi_n(\omega) \) and \( \{\alpha_{in}\} \) are real sequences in \([0, 1]\) for all \( n \in \mathbb{N} \).

We need the following two results for proving our main results.

**Lemma 1.6** ([20]). Let \( X \) be a separable metric space and \( Y \) be a metric space. If \( f : \Omega \times X \to Y \) is measurable in \( \omega \in \Omega \) and continuous in \( x \in X \), and if \( x : \Omega \to X \) is measurable, then \( f(\cdot, x(\cdot)) : \Omega \to Y \) is measurable.

**Lemma 1.7** ([13]). Let \( \{\beta_n\} \) and \( \{\gamma_n\} \) be sequences of nonnegative real numbers satisfying the following conditions:

\[
    \beta_{n+1} \leq (1 + \gamma_n) \beta_n, \quad \sum_{n=1}^{\infty} \gamma_n < \infty
\]

We have

(i) \( \lim_{n \to \infty} \beta_n \) exists;

(ii) if \( \liminf_{n \to \infty} \beta_n = 0 \), then \( \lim_{n \to \infty} \beta_n = 0 \).
2. Main results

In this section, we give some conditions for the convergence of the random iteration process (1.3) to a common random fixed point of a finite family asymptotically quasi-nonexpansive random mappings \( \{T_i, i \in J\} \). We first prove the following lemma.

Lemma 2.1. Let \( K \) be a nonempty closed convex subset of a separable complete convex metric space \( (X,d) \). Let \( \{T_i : i \in J\} : \Omega \times K \rightarrow K \) be a finite family of asymptotically quasi-nonexpansive random mappings with \( r_n(\omega) : \Omega \rightarrow [0,\infty) \) for each \( \omega \in \Omega \). Suppose that the sequence \( \{\xi_n(\omega)\} \) is defined as (1.3) and \( \sum_{n=1}^{\infty} \alpha_{kn} < \infty \). If \( F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset \), then

(i) there exists a constant \( M_0 > 0 \) such that

\[
 d(\xi_{n+1}(\omega), \xi(\omega)) \leq (1 + \alpha_{kn}M_0)d(\xi_n(\omega), \xi(\omega))
\]

for all \( \xi(\omega) \in F \) and \( n \in \mathbb{N} \);

(ii) there exists a constant \( M_1 > 0 \) such that

\[
 d(\xi_{n+m}(\omega), \xi(\omega)) \leq M_1d(\xi_n(\omega), \xi(\omega))
\]

for all \( \xi(\omega) \in F \) and \( n, m \in \mathbb{N} \).

Proof. (i) Since \( \{T_i : i \in J\} : \Omega \times K \rightarrow K \) be a finite family of asymptotically quasi-nonexpansive random mappings with \( r_n(\omega) : \Omega \rightarrow [0,\infty) \) for each \( \omega \in \Omega \), there exists a measurable mapping \( r_n(\omega) = \max\{r_{1n}(\omega), r_{2n}(\omega), \ldots, r_{kn}(\omega)\} \) for each \( \omega \in \Omega \) with \( \lim_{n \rightarrow \infty} r_n(\omega) = 0 \), such that

\[
 d(T^n_i(\omega, \eta(\omega)), \xi(\omega)) \leq (1 + r_n(\omega))d(\eta(\omega), \xi(\omega))
\]

where \( i \in J \) and \( \eta : \Omega \rightarrow K \) is any measurable mapping. By (1.3), we have

\[
 d(\eta_{1n}(\omega), \xi(\omega)) = d(W(T^n_1(\omega, \eta_{0n}(\omega)), \xi_n(\omega); \alpha_{1n}), \xi(\omega))
\]

\[
 \leq \alpha_{1n}d(T^n_1(\omega, \eta_{0n}(\omega)), \xi(\omega)) + (1 - \alpha_{1n})d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq \alpha_{1n}(1 + r_n(\omega))d(\xi_n(\omega), \xi(\omega)) + (1 - \alpha_{1n})d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq (1 + \alpha_{1n}(1 + r_n(\omega)))d(\xi_n(\omega), \xi(\omega)).
\]

Since \( r_n(\omega) : \Omega \rightarrow [0,\infty) \) and \( \lim_{n \rightarrow \infty} r_n(\omega) = 0 \), there exists a constant \( L > 0 \) such that

\[
 L = \sup_{n \geq 1} \{1 + r_n(\omega)\} < \infty.
\]

Therefore,

\[
 d(\eta_{1n}(\omega), \xi(\omega)) \leq (1 + L)d(\xi_n(\omega), \xi(\omega)).
\]

Assume that

\[
 d(\eta_{in}(\omega), \xi(\omega)) \leq (1 + L)^id(\xi_n(\omega), \xi(\omega))
\]

holds for some \( 1 \leq i \leq k - 1 \). Then

\[
 d(\eta_{i+1n}(\omega), \xi(\omega)) = d(W(T^n_{i+1}(\omega, \eta_{0n}(\omega)), \xi_n(\omega); \alpha_{(i+1)n}), \xi(\omega))
\]

\[
 \leq \alpha_{(i+1)n}d(T^n_{i+1}(\omega, \eta_{0n}(\omega)), \xi(\omega)) + (1 - \alpha_{(i+1)n})d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq \alpha_{(i+1)n}(1 + r_n(\omega))d(\eta_{0n}(\omega), \xi(\omega)) + (1 - \alpha_{(i+1)n})d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq (1 - \alpha_{(i+1)n} + \alpha_{(i+1)n}L(1 + L)^i)d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq (1 + L(1 + L)^i)d(\xi_n(\omega), \xi(\omega))
\]

\[
 \leq (1 + L)^{i+1}d(\xi_n(\omega), \xi(\omega))
\]
So, by induction, we obtain
\[ d(\eta_n(\omega), \xi(\omega)) \leq (1 + L)^i d(\xi_n(\omega), \xi(\omega)) \]
for all \(1 \leq i \leq k\). Now, by (1.3) and the above inequality, we get
\[
d(\xi_{n+1}(\omega), \xi(\omega)) = d(W(T^n_k(\omega, \eta_{(k-1)n}(\omega)), \xi_n(\omega); \alpha_{kn}), \xi(\omega)) \\
\leq \alpha_{kn} d(T_k(\omega, \eta_{(k-1)n}(\omega)), \xi(\omega)) + (1 - \alpha_{kn}) d(\xi_n(\omega), \xi(\omega)) \\
\leq \alpha_{kn} (1 + r_n(\omega)) d(\eta_{(k-1)n}(\omega), \xi(\omega)) + (1 - \alpha_{kn}) d(\xi_n(\omega), \xi(\omega)) \\
\leq (1 - \alpha_{kn} + \alpha_{kn} L (1 + L)^k) d(\xi_n(\omega), \xi(\omega)) \\
\leq (1 + \alpha_{kn} M_0) d(\xi_n(\omega), \xi(\omega))
\]
where \(M_0 = (1 + L)^k > 0\).

(ii) Notice that \(1 + x \leq e^x\) for all \(x \geq 0\). Using this and \(\sum_{n=1}^{\infty} \alpha_{kn} < \infty\), we have
\[
d(\xi_{n+m}(\omega), \xi(\omega)) \leq (1 + \alpha_{k(n+m-1)} M_0) d(\xi_{n+m-1}(\omega), \xi(\omega)) \\
\leq e^{\alpha_{k(n+m-1)} M_0} (1 + \alpha_{k(n+m-2)} M_0) d(\xi_{n+m-2}(\omega), \xi(\omega)) \\
\leq e^{\alpha_{k(n+m-1)} + \alpha_{k(n+m-2)} M_0} d(\xi_{n+m-2}(\omega), \xi(\omega)) \\
\ldots \\
\leq e^{M_0 \sum_{j=1}^{\infty} \alpha_{kj}} d(\xi_n(\omega), \xi(\omega)) \\
\leq M_1 d(\xi_n(\omega), \xi(\omega))
\]
where \(M_1 = e^{M_0 \sum_{j=1}^{\infty} \alpha_{kj}} > 0\).

\[\Box\]

**Theorem 2.2.** Let \(K\) be a nonempty closed convex subset of a separable complete convex metric space \((X, d)\) with a measurable convex structure \(W\). Let \(\{T_i : i \in J\} : \Omega \times K \to K\) be a finite family of continuous asymptotically quasi-nonexpansive random mappings with \(r_m(\omega) : \Omega \to [0, \infty)\) for each \(\omega \in \Omega\). Suppose that the sequence \(\{\xi_n(\omega)\}\) is defined as (1.3) and \(\sum_{n=1}^{\infty} \alpha_{kn} < \infty\). If \(F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset\), then \(\{\xi_n(\omega)\}\) converges to a common fixed point of \(\{T_i : i \in J\}\) if and only if \(\lim \inf_{n \to \infty} d(\xi_n(\omega), F) = 0\), where \(d(\xi_n(\omega), F) = \inf \{d(\xi_n(\omega), \eta(\omega)) : \forall \eta(\omega) \in F\}\) for each \(\omega \in \Omega\).

**Proof.** The necessity is obvious. Thus, we only need prove the sufficiency. From Lemma 2.1 (i), we have
\[ d(\xi_{n+1}(\omega), F) \leq (1 + \alpha_{kn} M_0) d(\xi_n(\omega), F). \]
By Lemma 1.7 and \(\sum_{n=1}^{\infty} \alpha_{kn} < \infty\), we know that
\[ \lim_{n \to \infty} d(\xi_n(\omega), F) \]
exists. Since \(\lim \inf_{n \to \infty} d(\xi_n(\omega), F) = 0\), we obtain
\[ \lim_{n \to \infty} d(\xi_n(\omega), F) = 0 \]
for each \(\omega \in \Omega\).

Next, We show that \(\{\xi_n(\omega)\}\) is a Cauchy sequence. Indeed, for any \(\varepsilon > 0\), there exists a constant \(N_0\) such that for all \(n \geq N_0\), we have
\[ d(\xi_n(\omega), F) \leq \frac{\varepsilon}{2M_1}. \]
In particular, there exist a \(p_1(\omega) \in F\) and a constant \(N_1 > N_0\) such that
\[ d(\xi_{N_1}(\omega), p_1(\omega)) \leq \frac{\varepsilon}{2M_1}. \]
It follows from Lemma 2.1(ii) that for $n > N_1$, we have
\[
d(\xi_{n+m}(\omega),\xi_n(\omega)) \leq d(\xi_{n+m}(\omega),p_1(\omega)) + d(p_1(\omega),\xi_n(\omega))
\leq M_1 d(\xi_{N_1}(\omega),p_1(\omega)) + M_1 d(\xi_{N_1}(\omega),p_1(\omega))
\leq 2M_1 \frac{\varepsilon}{2M_1} = \varepsilon.
\]
This implies that \{\xi_n\} is a Cauchy sequence in a closed convex subset of a complete convex metric space. Therefore, \{\xi_n(\omega)\} converges to a point in $K$.

Suppose $\lim_{n \to \infty} \xi_n(\omega) = p(\omega)$ for each $\omega \in \Omega$. Since $T_i$ are continuous, by Lemma 1.6, we know that for any measurable mapping $f : \Omega \to K$, $T_i^p(\omega,f(\omega)) : \Omega \to K$ are measurable mappings. Thus, \{\xi_n(\omega)\} is a sequence of measurable mappings. Hence, $p(\omega) : \Omega \to K$ is also measurable. Notice that
\[
d(\xi_n(\omega),F) \leq d(\xi_n(\omega),p(\omega)) + d(p(\omega),F),
\]
and
\[
\lim_{n \to \infty} d(\xi_n(\omega),F) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(\xi_n(\omega),p(\omega)) = 0,
\]
we can conclude that $d(p(\omega),F) = 0$. Therefore, $p(\omega) \in F$.

\[
\square
\]

Remark 2.3. (i) Theorem 2.2 extends the corresponding results in \cite{1,2,4,6,8,11,12,13,14,17,16} to the convex metric space, which is a more general space;

(ii) Theorem 2.2 extends the corresponding results in \cite{4,6,10,20,22} to a finite family of asymptotically quasi-nonexpansive random mappings, which are stochastic generalizations of asymptotically quasi-nonexpansive mappings;

(iii) In Theorem 2.2, we remove the condition: “$\sum_{n=1}^{\infty} \alpha_{in} < \infty$, $i \in J$”, which is required in many other papers (see, e.g., \cite{1,2,9,10,16,20,22}). And the condition “$\sum_{n=1}^{\infty} \alpha_{in} < \infty$, $i \in J$” is replaced with “$\sum_{n=1}^{\infty} \alpha_{kn} < \infty$.”

By Remark 1.3, we can get the following result:

Corollary 2.4. Let $K$ be a nonempty closed convex subset of a separable complete convex metric space $(X,d)$ with a measurable convex structure $W$. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of asymptotically nonexpansive random mappings with $r_{in}(\omega) : \Omega \to [0,\infty)$ for each $\omega \in \Omega$. Suppose that the sequence \{\xi_n(\omega)\} is defined as \cite{1.3} and $\sum_{n=1}^{\infty} \alpha_{kn} < \infty$. If $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, then \{\xi_n(\omega)\} converges to a common fixed point of $\{T_i : i \in J\}$ if and only if $\lim_{n \to \infty} d(\xi_n(\omega),F) = 0$, where $d(\xi_n(\omega),F) = \inf\{d(\xi_n(\omega),\eta(\omega)) : \forall \eta(\omega) \in F\}$ for each $\omega \in \Omega$.

Theorem 2.5. Let $K$ be a nonempty closed convex subset of a separable complete convex metric space $(X,d)$ with a measurable convex structure $W$. Let $\{T_i : i \in J\} : \Omega \times K \to K$ be a finite family of continuous asymptotically quasi-nonexpansive random mappings with $r_{in}(\omega) : \Omega \to [0,\infty)$ for each $\omega \in \Omega$. Suppose that the sequence \{\xi_n(\omega)\} is defined as \cite{1.3}, $\sum_{n=1}^{\infty} \alpha_{kn} < \infty$ and $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. If for some given $1 \leq l \leq k$ and each $\omega \in \Omega$,

(i) $\lim_{n \to \infty} d(T_l(\omega,\xi_n(\omega)),\xi_n(\omega)) = 0$,

(ii) there exists a constant $M_2 > 0$ such that

\[
d(T_l(\omega,\xi_n(\omega)),\xi_n(\omega)) \geq M_2 d(\xi_n(\omega),F).
\]

Then \{\xi_n(\omega)\} converges to a common fixed point of $\{T_i : i \in J\}$.
Proof. From the conditions (i) and (ii), we have
\[ \lim_{n \to \infty} d(\xi_n(\omega), F) = 0. \]
Therefore, from the proof of Theorem 2.2, we know that \( \{\xi_n(\omega)\} \) converges to a common fixed point of \( \{T_i : i \in J\} \) \( \Box \)

**Theorem 2.6.** Let \( K \) be a nonempty closed convex subset of a separable complete convex metric space \((X, d)\) with a measurable convex structure \(W\). Let \( \{T_i : i \in J\} : \Omega \times K \to K \) be a finite family of continuous asymptotically quasi-nonexpansive random mappings with \( r_{i,n}(\omega) : \Omega \to [0, \infty) \) for each \( \omega \in \Omega \). Suppose that the sequence \( \{\xi_n(\omega)\} \) is defined as \([1.3]\), \( \sum_{n=1}^{\infty} \alpha_{kn} < \infty \) and \( F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset \). If

(i) for all \( 1 \leq i \leq k \) and each \( \omega \in \Omega \), \( \lim_{n \to \infty} d(T_i(\omega, \xi_n(\omega)), \xi_n(\omega)) = 0 \);

(ii) for some \( 1 \leq l' \leq k \), \( T_{l'} \) is semicompact.

Then \( \{\xi_n(\omega)\} \) converges to a common fixed point of \( \{T_i : i \in J\} \).

Proof. Since \( T_{l'} \) is semicompact and \( \lim_{n \to \infty} d(T_{l'}(\omega, \xi_n(\omega)), \xi_n(\omega)) = 0 \), there exists a subsequence \( \{\xi_{nj}(\omega)\} \subset \{\xi_n(\omega)\} \) such that \( \lim_{j \to \infty} \xi_{nj}(\omega) = \xi'(\omega) \) for each \( \omega \in \Omega \). Since \( T_i \) are continuous, it follows that \( \{\xi_n\} \) is a sequence of measurable mappings. Therefore, \( \xi'(\omega) : \Omega \to K \) is also measurable. Hence, it follows from
\[ d(T_i(\omega, \xi'(\omega)), \xi'(\omega)) = \lim_{n \to \infty} d(T_i(\omega, \xi_{nj}(\omega)), \xi_{nj}(\omega)) = 0 \]
that \( \xi'(\omega) \in F \). By Lemma 2.1 (i), we have
\[ d(\xi_{n+1}(\omega), \xi'(\omega)) \leq (1 + \alpha_{kn} M_0) d(\xi_n(\omega), \xi'(\omega)). \]
According to Lemma 1.7 and \( \sum_{n=1}^{\infty} \alpha_{kn} < \infty \), there exists a constant \( \delta \geq 0 \) such that
\[ \lim_{n \to \infty} d(\xi_n(\omega), \xi'(\omega)) = \delta. \]
Since \( \lim_{j \to \infty} \xi_{nj}(\omega) = \xi'(\omega) \), we have \( \delta = 0 \). Therefore, \( \{\xi_n(\omega)\} \) converges to a common fixed point of \( \{T_i : i \in J\} \) \( \Box \)

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