COEFFICIENT ESTIMATES FOR A CLASS OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we introduce and investigate an interesting subclass $B^{h,p}_\Sigma$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $B^{h,p}_\Sigma$ we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent work of Brannan and Taha [1].

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Let $A$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

We also denote by $S$ the class of all functions in the normalized analytic function class $A$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [2] ensures that the image of $\mathbb{U}$ under every univalent function $f \in S$ contains a disk of radius $1/4$. Thus every function $f \in A$ has an inverse $f^{-1}$, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f) ; \quad r_0(f) \geq \frac{1}{4}\right).$$

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In fact, the inverse function \( f^{-1} \) is given by
\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots.
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by \((\ref{eq:bi-univalent})\). For a brief history and interesting examples of functions in the class \( \Sigma \), see \cite{3}.

Brannan and Taha \cite{11} introduced the following two subclasses of the bi-univalent function class \( \Sigma \) and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) of functions in each of these subclasses.

**Definition 1.** (see \cite{11}) A function \( f(z) \) given by \((\ref{eq:bi-univalent})\) is said to be in the class \( S_\Sigma^* [\alpha] \) \( (0 < \alpha \leq 1) \) if the following conditions are satisfied:

\[
(1.2) \quad f \in \Sigma \quad \text{and} \quad \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (z \in U)
\]

and

\[
(1.3) \quad \left| \arg \left( \frac{wg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (w \in U),
\]

where the function \( g \) is given by
\[
(1.4) \quad g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots.
\]

We call \( S_\Sigma^* [\alpha] \) the class of strongly bi-starlike functions of order \( \alpha \).

**Theorem 1.1.** (see \cite{11}) Let the function \( f(z) \) given by \((\ref{eq:bi-univalent})\) be in the class \( S_\Sigma^* [\alpha] \) \( (0 < \alpha \leq 1) \). Then

\[
(1.5) \quad |a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}
\]

and

\[
(1.6) \quad |a_3| \leq 2\alpha.
\]

**Definition 2.** (see \cite{11}) A function \( f(z) \) given by \((\ref{eq:bi-univalent})\) is said to be in the class \( S_\Sigma^* (\beta) \) \( (0 \leq \beta < 1) \) if the following conditions are satisfied:

\[
(1.7) \quad f \in \Sigma \quad \text{and} \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (z \in U)
\]

and

\[
(1.8) \quad \Re \left\{ \frac{wg'(w)}{g(w)} \right\} > \beta \quad (w \in U),
\]

where the function \( g \) is defined by \((\ref{eq:strongly-bi-starlike})\). We call \( S_\Sigma^* (\beta) \) the class of bi-starlike functions of order \( \beta \).
Theorem 1.2. (see [3]) Let the function $f(z)$, given by the Taylor-Maclaurin series expansion (1.1), be in the class $S^*_\Sigma(\beta)$ ($0 \leq \beta < 1$). Then

\begin{equation}
|a_2| \leq \sqrt{2(1-\beta)}
\end{equation}

and

\begin{equation}
|a_3| \leq 2(1-\beta).
\end{equation}

Here, in our present sequel to some of the aforecited works (especially [3]), we introduce the following subclass of the analytic function class $A$, analogously to the definition given by Xu et al. [4].

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min \{ \Re(h(z)), \Re(p(z)) \} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$ 

Also let the function $f$, defined by (1.1), be in the analytic function class $A$. We say that $f \in B_{h,p}^h$ if the following conditions are satisfied:

\begin{equation}
f \in \Sigma \quad \text{and} \quad \frac{zf'(z)}{f(z)} \in h(\mathbb{U}) \quad (z \in \mathbb{U})
\end{equation}

and

\begin{equation}
\frac{wg'(w)}{g(w)} \in p(\mathbb{U}) \quad (w \in \mathbb{U}),
\end{equation}

where the function $g$ is defined by (1.4).

Remark 1. There are many choices of the functions $h$ and $p$ which would provide interesting subclasses of the analytic function class $A$. For example, if we let

$$h(z) = \left(\frac{1+z}{1-z}\right)^\alpha \quad \text{and} \quad p(z) = \left(\frac{1-z}{1+z}\right)^\alpha \quad (0 < \alpha \leq 1, z \in \mathbb{U})$$

or

$$h(z) = \frac{1+(1-2\beta)z}{1-z} \quad \text{and} \quad p(z) = \frac{1-(1-2\beta)z}{1+z} \quad (0 \leq \beta < 1, z \in \mathbb{U}),$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in B_{h,p}^{h,p}$, then

$$f \in \Sigma \quad \text{and} \quad \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, z \in \mathbb{U})$$

and

$$\left| \arg\left(\frac{wg'(w)}{g(w)}\right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, w \in \mathbb{U})$$

or

$$f \in \Sigma \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (0 \leq \beta < 1, z \in \mathbb{U})$$
and
\[ \Re \left( \frac{w g'(w)}{g(w)} \right) > \beta \quad (0 \leq \beta < 1, \ w \in U) , \]
where the function \( g \) is defined by (1.2). This means that
\[ f \in S^*_\Sigma [\alpha] \quad (0 < \alpha \leq 1) \]
or
\[ f \in S^*_\Sigma (\beta) \quad (0 \leq \beta < 1) . \]

Motivated and stimulated especially by the work of Brannan and Taha \[1\],
we propose to investigate the bi-univalent function class \( B_{h;p} \)
introduced here in Definition 3 and derive coefficient estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) for a function \( f \in B_{h;p} \) given by (1.1). Our results for the bi-univalent function class \( B_{h;p} \) would generalize and improve the related work of Brannan and Taha \[1\].

2. A Set of General Coefficient Estimates

In this section we state and prove our general results involving the bi-univalent function class \( B_{h;p} \) given by Definition 3.

Theorem 2.1. Let the function \( f(z) \) given by the Taylor-Maclaurin series expansion (1.1), be in the bi-univalent function class \( B_{h;p} \). Then

\[ |a_2| \leq \min \left\{ \frac{\sqrt{|h'(0)|^2 + |p'(0)|^2}}{2}, \frac{\sqrt{|h''(0)| + |p''(0)|}}{4} \right\} \] (2.1)

and

\[ |a_3| \leq \min \left\{ \frac{|h'(0)|^2 + |p'(0)|^2}{2} + \frac{|h''(0)| + |p''(0)|}{8}, \frac{3|h''(0)| + |p''(0)|}{8} \right\} . \] (2.2)

Proof. First of all, we write the argument inequalities in (1.11) and (1.12) in their equivalent forms as follows:

\[ \frac{zf'(z)}{f(z)} = h(z) \quad (z \in U) , \]

and

\[ \frac{wg'(w)}{g(w)} = p(w) \quad (w \in U) , \]

respectively, where \( h(z) \) and \( p(w) \) satisfy the conditions of Definition 3. Furthermore, the functions \( h(z) \) and \( p(w) \) have the following Taylor-Maclaurin series expansions:

\[ h(z) = 1 + h_1z + h_2z^2 + \cdots \]
and
\[ p(w) = 1 + p_1 w + p_2 w^2 + \cdots, \]
respectively. Now, upon equating the coefficients of \( \frac{zf'(z)}{f(z)} \) with those of \( h(z) \) and the coefficients of \( \frac{wg'(w)}{g(w)} \) with those of \( p(w) \), we get

\begin{align*}
(2.3) & \quad a_2 = h_1, \\
(2.4) & \quad 2a_3 - a_2^2 = h_2, \\
(2.5) & \quad -a_2 = p_1 \\
\end{align*}
and

\begin{align*}
(2.6) & \quad 3a_2^2 - 2a_3 = p_2.
\end{align*}

From (2.3) and (2.3), we obtain

\begin{align*}
(2.7) & \quad h_1 = -p_1 \\
\end{align*}
and

\begin{align*}
(2.8) & \quad 2a_2^2 = h_1^2 + p_1^2.
\end{align*}

Also, from (2.4) and (2.5), we find that

\begin{align*}
(2.9) & \quad 2a_2^2 = h_2 + p_2.
\end{align*}

Therefore, we find from the equations (2.3) and (2.3) that

\[ |a_2|^2 \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2} \]

and

\[ |a_2|^2 \leq \frac{|h''(0)| + |p''(0)|}{4}, \]

respectively. So we get the desired estimate on the coefficient \(|a_2|\) as asserted in (2.1).

Next, in order to find the bound on the coefficient \(|a_3|\), we subtract (2.10) from (2.4). We thus get

\begin{align*}
(2.10) & \quad 4a_3 - 4a_2^2 = h_2 - p_2.
\end{align*}

Upon substituting the value of \( a_2^2 \) from (2.8) into (2.10), it follows that

\[ a_3 = \frac{h_1^2 + p_1^2}{2} + \frac{h_2 - p_2}{4}. \]

We thus find that

\[ |a_3| \leq \frac{|h'(0)|^2 + |p'(0)|^2}{2} + \frac{|h''(0)| + |p''(0)|}{8}. \]
On the other hand, upon substituting the value of $a_2^2$ from (2.9) into (2.10), it follows that

$$a_3 = \frac{3h_2 + p_2}{4}.$$ 

We thus obtain

$$|a_3| \leq \frac{3|h''(0)| + |p''(0)|}{8}.$$ 

This evidently completes the proof of Theorem 2.1. \qed

3. Corollaries and Consequences

If we set

$$h(z) = \left(\frac{1 + z}{1 - z}\right)^{\alpha} \quad \text{and} \quad p(z) = \left(\frac{1 - z}{1 + z}\right)^{\alpha} \quad (0 < \alpha \leq 1, \ z \in \mathbb{U})$$

in Theorem 2.1, we can readily deduce the following corollary.

**Corollary 3.1.** Let the function $f(z)$, given by the Taylor-Maclaurin series expansion (1.1), be in the bi-univalent function class $S^*_\Sigma[\alpha]$ $(0 < \alpha \leq 1)$. Then

$$|a_2| \leq \sqrt{2\alpha}$$

and

$$|a_3| \leq 2\alpha^2.$$

**Remark 2.** It is easy to see that

$$\sqrt{2\alpha} \leq \frac{2\alpha}{\sqrt{1 + \alpha}} \quad (0 < \alpha \leq 1)$$

and

$$2\alpha^2 \leq 2\alpha \quad (0 < \alpha \leq 1),$$

which, in conjunction with Corollary 3.1, would obviously yield an improvement of Theorem 2.1.

If we set

$$h(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{and} \quad p(z) = \frac{1 - (1 - 2\beta)z}{1 + z} \quad (0 \leq \beta < 1, \ z \in \mathbb{U})$$

in Theorem 2.1, we can readily deduce the following corollary.

**Corollary 3.2.** Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $S^*_\Sigma(\beta)$ $(0 \leq \beta < 1)$. Then

$$|a_2| \leq \sqrt{2(1 - \beta)}$$

and

$$|a_3| \leq \left\{\begin{array}{ll}
2(1 - \beta), & 0 \leq \beta \leq \frac{3}{4} \\
\frac{4(1 - \beta)^2 + (1 - \beta)}{2}, & \frac{3}{4} \leq \beta < 1.
\end{array}\right.$$
Remark 3. It is easy to see that
(i) if $0 \leq \beta \leq \frac{3}{4}$, then
$$|a_3| \leq 2 (1 - \beta);$$
(ii) if $\frac{3}{4} \leq \beta < 1$, then
$$|a_3| \leq 4 (1 - \beta)^2 + (1 - \beta) \leq 2 (1 - \beta).$$
Thus, clearly, Corollary 3 is an improvement of Theorem 2.

References


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