MIXED PROBLEM WITH INTEGRAL CONDITIONS FOR A CERTAIN CLASS OF HYPERBOLIC EQUATIONS

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We study a mixed problem with purely integral conditions for a class of two-dimensional second-order hyperbolic equations. We prove the existence, uniqueness, and the continuous dependence upon the data of a generalized solution. We use a functional analysis method based on a priori estimate and on the density of the range of the operator generated by the considered problem.

1. Introduction

The present paper is devoted to the proof of existence and uniqueness of a generalized solution for a mixed problem with only integral conditions related to a certain class of second-order hyperbolic equations in a two-dimensional structure. That is, we consider the problem of searching a function \( u = u(x,t) \), solution of the problem

\[
L u = u_{tt} - a(t) \Delta u = f(x,t), \quad x = (x_1, x_2) \in \Omega, \quad t \in (0,T),
\]

where \( \Omega = (0,a) \times (0,b_i) \) and \( b_i, T, i = 1,2 \), are known constants and \( a(t) \) is a given function satisfying the conditions

\[
c_0 \leq a(t) \leq c_1, \quad a'(t) \leq c_2,
\]

where \( c_1, c_2 \) are positive constants.

To (1.1), we associate the initial conditions

\[
\ell_1 u = u(x,0) = \varphi(x), \quad \ell_2 u = u_t(x,0) = \beta(x), \quad x \in \Omega,
\]

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and the integral conditions

\[ \int_0^{b_i} x_i^k i_u(x,t) \, dx = 0, \quad i = 1, 2; \quad k = 0, 1, \quad (1.4) \]

where \( f, \varphi, \) and \( \beta \) are given functions such that \( f \in C(\bar{Q}) \) and \( \varphi, \beta \in C^1(\bar{\Omega}). \)

The given data satisfy the consistency conditions

\[ \int_0^{b_i} x_i^k \varphi \, dx = \int_0^{b_i} x_i^k \beta \, dx = 0, \quad i = 1, 2; \quad k = 0, 1. \quad (1.5) \]

The results concerning problems with integral conditions related to one-dimensional parabolic equations are due to Batten [1], Cannon [7, 8], Cannon and van der Hoek [10, 11], Cannon et al. [9], Kamynin [13], Ionkin [12], Yurchuk [17], Benouar and Yurchuk [2], Muravey-Philinovskii [14], Shi [16], Bouziani [3, 4], and Bouziani and Benouar [6]. For problems related to one-dimensional hyperbolic equations we have the result of Bouziani [5], in which a Neumann and an integral condition are combined.

The present paper can be considered as an extension of Bouziani [5] in the way that the conditions are purely integral and the considered equation is a two-dimensional one. We first write the posed problem in its operational form \( Lu = F, \) where the operator \( L \) is considered from the Banach space \( E \) into the Hilbert space \( F, \) which are conveniently chosen, then we establish an energy inequality for the operator \( L, \) and extend the obtained estimate to the closure \( \bar{L} \) of the operator \( L. \) Finally, we prove the density of the range \( \mathcal{R}(L) \) of the operator \( L \) in the space \( F. \)

2. Energy inequality and its consequences

Problem (1.1), (1.3), and (1.4) can be considered as the resolution of the operator equation

\[ Lu = F, \quad (2.1) \]

where \( L = (\mathcal{L}, \ell_1, \ell_2), \) \( F = (f, \varphi, \beta) \) and \( L \) is an operator defined on \( E \) into \( F, \) where \( E \) is the Banach space of functions \( J_{x_1 x_2} u \in L^2(Q), \) having the finite norm

\[ \|u\|^2_E = \sup_{0 \leq \tau \leq 0} \int_{\Omega} \left( (J_{x_1} u(\cdot, \cdot, \tau))^2 + (J_{x_2} u(\cdot, \cdot, \tau))^2 + (J_{x_1 x_2} u_t(\cdot, \cdot, \tau))^2 \right) \, dx_1 \, dx_2 \quad (2.2) \]

with \( J_{x_1} u = \int_0^{x_1} u(\xi, x_2, t) \, d\xi, \ J_{x_1 x_2} u = \int_0^{x_1} \int_0^{x_2} u(\xi, \eta, t) \, d\xi \, d\eta, \) and \( F \) is the
Hilbert space equipped with the scalar product
\[
(\langle \mathcal{L} u, \ell_1 u, \ell_2 u \rangle, (f, \varphi, \beta))_F
= \int_Q \mathcal{I}_{x_1x_2} \langle \mathcal{L} u \rangle \cdot \mathcal{I}_{x_1x_2} f \, dx \, dt + \int_{\Omega} \mathcal{I}_{x_1} \ell_1 u \cdot \mathcal{I}_{x_1} \varphi \, dx
+ \int_{\Omega} \mathcal{I}_{x_2} \ell_1 u \cdot \mathcal{I}_{x_2} \varphi \, dx + \int_{\Omega} \mathcal{I}_{x_1x_2} \ell_2 u \cdot \mathcal{I}_{x_1x_2} \beta \, dx,
\]
(2.3)
and the associated norm
\[
\| \mathcal{L} u \|_F^2 = \int_Q (\mathcal{I}_{x_1x_2} (\mathcal{L} u))^2 \, dx \, dt
+ \int_{\Omega} \left( (\mathcal{I}_{x_1} \ell_1 u)^2 + (\mathcal{I}_{x_2} \ell_1 u)^2 + (\mathcal{I}_{x_1x_2} \ell_2 u)^2 \right) \, dx.
\]
(2.4)

The domain of definition \( D(\mathcal{L}) \) of the operator \( \mathcal{L} \) is the set of functions \( \mathcal{I}_{x_1x_2} u \in L^2(Q) \) such that \( \mathcal{I}_{x_1x_2} u_t, \mathcal{I}_{x_1x_2} u_{x_1x_1}, \mathcal{I}_{x_1x_2} u_{x_2x_2} \in L^2(Q) \), and the conditions (1.4) are fulfilled.

**Theorem 2.1.** If \( a(t) \) satisfies conditions (1.2), then for all functions \( u \in D(\mathcal{L}) \) we have the a priori estimate
\[
\| u \|_E \leq c \| \mathcal{L} u \|_F,
\]
(2.5)
where \( c \) is a positive constant independent of the solution \( u \).

**Proof.** We consider the scalar product in \( L^2(Q^\tau) \) of (1.1) and the integro-differential operator
\[
\mathcal{M} u = \mathcal{I}^2_{x_1x_2} u_t = \int_0^x \int_0^x \int_0^\ell_2 \int_0^{\ell_1} u_t(\eta_1, \eta_2, t) \, d\eta_2 \, d\eta_1 \, d\xi_2 \, d\xi_1, \tag{2.6}
\]
where \( Q^\tau = \Omega \times (0, \tau) \) and \( \tau \in (0, T) \), we obtain
\[
\int_{Q^\tau} u_{tt} \cdot \mathcal{I}^2_{x_1x_2} u_t \, dx \, dt
- \int_{Q^\tau} a(t) u_{x_1x_1} \cdot \mathcal{I}^2_{x_1x_2} u_t \, dx \, dt
- \int_{Q^\tau} a(t) u_{x_2x_2} \cdot \mathcal{I}^2_{x_1x_2} u_t \, dx \, dt
= \int_{Q^\tau} f \cdot \mathcal{I}^2_{x_1x_2} u_t \, dx \, dt.
\]
(2.7)
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We separately consider the integrals of the equality (2.7). Integrating by parts and taking into account conditions (1.3) and (1.4), we get

\[
\int_{Q^\tau} u_{tt} \mathcal{J}_{x_1x_2}^2 u_t \, dx \, dt
\]
\[
= \frac{1}{2} \int_{Q^\tau} (\mathcal{J}_{x_1x_2}^2 u_t (\xi_1, \xi_2, \tau))^2 \, dx - \frac{1}{2} \int_{Q^\tau} (\mathcal{J}_{x_1x_2}^2 u_t (\xi_1, \xi_2, \tau))^2 \, dx,
\]
\[
- \int_{Q^\tau} a(t) u_{x_1x_1} \mathcal{J}_{x_1x_2}^2 u_t \, dx \, dt
\]
\[
= \frac{1}{2} \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1x_2} u (x_1, \xi_2, \tau))^2 \, dx - \frac{1}{2} \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1x_2} u (x_1, \xi_2, \tau))^2 \, dx \, dt,
\]
\[
- \int_{Q^\tau} a(t) u_{x_2x_2} \mathcal{J}_{x_1x_2}^2 u_t \, dx \, dt
\]
\[
= \frac{1}{2} \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1x_2} u (\xi_1, x_2, \tau))^2 \, dx - \frac{1}{2} \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1x_2} u (\xi_1, x_2, \tau))^2 \, dx \, dt,
\]
\[
\int_{Q^\tau} f \mathcal{J}_{x_1x_2}^2 u_t \, dx \, dt = \int_{Q^\tau} \mathcal{J}_{x_1x_2} f \mathcal{J}_{x_1x_2} u_t \, dx \, dt.
\] (2.11)

Substitution of (2.8), (2.9), (2.10), and (2.11) into (2.7) yields

\[
\int_{Q^\tau} (\mathcal{J}_{x_1x_2} u_t (\xi_1, \xi_2, \tau))^2 \, dx + \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1} u (x_1, \xi_2, \tau))^2 \, dx
\]
\[
+ \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1} u (\xi_1, x_2, \tau))^2 \, dx
\]
\[
= 2 \int_{Q^\tau} \mathcal{J}_{x_1x_2} f \mathcal{J}_{x_1x_2} u_t \, dx \, dt + \int_{Q^\tau} a(0) (\mathcal{J}_{x_1} \varphi)^2 \, dx
\]
\[
+ \int_{Q^\tau} a(0) (\mathcal{J}_{x_1} \varphi)^2 \, dx + \int_{Q^\tau} (\mathcal{J}_{x_1x_2} \beta)^2 \, dx
\]
\[
+ \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1} u (x_1, \xi_2, \tau))^2 \, dx \, dt + \int_{Q^\tau} a(\tau) (\mathcal{J}_{x_1} u (\xi_1, x_2, \tau))^2 \, dx \, dt.
\] (2.12)

Using the Cauchy inequality and taking into account conditions (1.2), it follows that
\[ \| \mathcal{I} x_1, u(\xi_1, x_2, \tau) \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_2, u(x_1, \xi_2, \tau) \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_1 x_2 u_{t}(\xi_1, \xi_2, \tau) \|_{L^2(\Omega)}^2 \leq c_3 \left( \| \mathcal{I} x_1, x_2 f \|_{L^2(Q)}^2 + \| \mathcal{I} x_1 \varphi \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_2 \varphi \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_1 x_2 \beta \|_{L^2(\Omega)}^2 \right) + c_4 \int_0^\tau \left( \| \mathcal{I} x_1, u \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_2, u \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_1 x_2 u_{t} \|_{L^2(\Omega)}^2 \right) \, dt, \tag{2.13} \]

where
\[ c_3 = \max \left( \frac{1}{c_0}, c_1 \right), \quad c_4 = \max \left( \frac{1}{c_0}, c_2 \right). \tag{2.14} \]

Applying the Gronwall's lemma to inequality (2.13), we get
\[ \| \mathcal{I} x_1, u(\xi_1, x_2, \tau) \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_2, u(x_1, \xi_2, \tau) \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_1 x_2 u_{t}(\xi_1, \xi_2, \tau) \|_{L^2(\Omega)}^2 \leq c_3 e^{c_4 \tau} \left( \| \mathcal{I} x_1, x_2 f \|_{L^2(Q)}^2 + \| \mathcal{I} x_1 \varphi \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_2 \varphi \|_{L^2(\Omega)}^2 + \| \mathcal{I} x_1 x_2 \beta \|_{L^2(\Omega)}^2 \right). \tag{2.15} \]

Since the right-hand side of (2.15) does not depend on \( \tau \), then by taking the supremum with respect to \( \tau \) over the interval \([0, T]\), we obtain the desired inequality (2.5), with \( c = c_3 / 2 \exp(c_4 T/2) \). This completes the proof of Theorem 2.1. \( \Box \)

Proposition 2.2. The operator \( L : E \to F \) is closable.

Proof. The proof of this proposition is analogous to Proposition 3.1 in [4]. \( \Box \)

Let \( \bar{L} \) be the closure of the operator \( L \), and \( D(\bar{L}) \) its domain of definition.

Definition 2.3. The solution of the equation
\[ \bar{L} u = \mathcal{F} \tag{2.16} \]
is called strong solution of problem (1.1), (1.3), and (1.4).

We extend inequality (2.5) to the set of solutions \( u \in D(\bar{L}) \) by passing to the limit and thus establish uniqueness of a strong solution and closedness of the range \( R(\bar{L}) \) of the operator \( L \) in the space \( F \).

3. Solvability of the problem

Theorem 3.1. If conditions (1.2) are satisfied, then for all \( \mathcal{F} = (f, \varphi, \beta) \in F \), there exists a unique strong solution \( u = \bar{L}^{-1} \mathcal{F} = \bar{L}^{-1} \mathcal{F} \) of problem (1.1), (1.3), and (1.4).
Proof. To prove that problem (1.1), (1.3), and (1.4) has a unique strong solution for all \( F \in F \), it suffices to prove that \( R(L) \) is dense in \( F \). For this we need the following proposition.

Proposition 3.2. If conditions (1.2) are satisfied, and if for \( J_{x_1 x_2} \omega \in L^2(Q) \),

\[
\int_Q J_{x_1 x_2} (Lu) \cdot J_{x_1 x_2} \omega \, dx \, dt = 0, \tag{3.1}
\]

for all the functions \( u \in D_0(L) = \{u/u \in D(L), \, \ell_1 u = \ell_2 u = 0\} \), then \( J_{x_1 x_2} \omega = 0 \) almost everywhere in \( Q \).

Using the fact that relation (3.1) is given for all \( u \in D_0(L) \), we can express it in a particular form.

Let \( u \) be defined as

\[
u(t, \tau) = \begin{cases} 
0, & 0 \leq t \leq s, \\
\int_s^t (t - \tau)u_{\tau\tau} \, d\tau, & s \leq t \leq T,
\end{cases} \tag{3.2}
\]

and let \( u_{tt} \) be the solution of the equation

\[
a(t)J_{x_1 x_2} u_{tt} = \int_t^T J_{x_1 x_2} \omega \, d\tau. \tag{3.3}
\]

We now have

\[
J_{x_1 x_2} \omega = -(a(t)J_{x_1 x_2} u_{tt})_t. \tag{3.4}
\]

To continue the proof of the proposition, we need the following lemma.

Lemma 3.3. If conditions (1.2) are satisfied, then the function \( u \) defined by relations (3.2) and (3.3) possesses derivatives with respect to \( t \) up to the third order belonging to \( L^2(Q) \).

The proof of this lemma is analogous to that of [3, Lemma 4.1].

We now prove the proposition. Replacing \( J_{x_1 x_2} \omega \) in (3.1) by its representation (3.4), we have

\[
-\int_Q J_{x_1 x_2} u_{tt} (a(t)J_{x_1 x_2} u_{tt})_t \, dx \, dt \\
+ \int_Q J_{x_1 x_2} u_{x_1 x_1} (a(t)J_{x_1 x_2} u_{tt})_t \, dx \, dt \\
+ \int_Q J_{x_1 x_2} u_{x_2 x_2} (a(t)J_{x_1 x_2} u_{tt})_t \, dx \, dt = 0. \tag{3.5}
\]
We write the terms of (3.5) in the form

\[
- \int_{\Omega} J_{x_1 x_2} u_{tt}(a(t) J_{x_1 x_2} u_{tt})_t \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} a(s) (J_{x_1 x_2} u_{tt}(x,s))^2 \, dx - \int_{Q_s} a'(t)(J_{x_1 x_2} u_{tt})^2 \, dx \, dt, \tag{3.6}
\]

\[
\int_{\Omega} J_{x_1 x_2} u_{x_1 x_1} (a(t) J_{x_1 x_2} u_{tt})_t \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} a(T) (J_{x_1} u_T(x,T))^2 \, dx - \frac{1}{2} \int_{Q_s} a'(t)(J_{x_1} u_T)^2 \, dx \, dt - \int_{Q_s} a'(t) J_{x_1} u J_{x_2} u_{tt} \, dx \, dt, \tag{3.7}
\]

\[
\int_{\Omega} J_{x_1 x_2} u_{x_2 x_2} (a(t) J_{x_1 x_2} u_{tt})_t \, dx \, dt \\
= \frac{1}{2} \int_{\Omega} a(T) (J_{x_1} u_T(x,T))^2 \, dx - \frac{1}{2} \int_{Q_s} a'(t)(J_{x_1} u_T)^2 \, dx \, dt - \int_{Q_s} a'(t) J_{x_1} u J_{x_2} u_{tt} \, dx \, dt. \tag{3.8}
\]

Combining conditions (3.5), (3.6), (3.7), and (3.8) and using conditions (1.2), we obtain the inequality

\[
\|J_{x_1 x_2} u_{tt}(x,s)\|_{L^2(\Omega)}^2 + \|J_{x_1} u_{tt}(x,T)\|_{L^2(\Omega)}^2 + \|J_{x_2} u_{tt}(x,T)\|_{L^2(\Omega)}^2 \\
\leq c_5 \left\{ \|J_{x_1 x_2} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_1} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_2} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_1} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_2} u_{tt}\|_{L^2(Q_s)}^2 \right\}, \tag{3.9}
\]

where

\[
\frac{c_5}{2} = \max \left( \frac{c_2}{2} + \frac{c_3}{2}, 1 \right). \tag{3.10}
\]

Using now the Friedrichs inequality [15], to express the norms of $J_{x_1} u$ and $J_{x_2} u$, in terms of the norms of $J_{x_1} u_t$ and $J_{x_2} u_t$, respectively, then it follows from (3.9) that

\[
\|J_{x_1 x_2} u_{tt}(x,s)\|_{L^2(\Omega)}^2 + \|J_{x_1} u_{tt}(x,T)\|_{L^2(\Omega)}^2 + \|J_{x_2} u_{tt}(x,T)\|_{L^2(\Omega)}^2 \\
\leq c_6 \left\{ \|J_{x_1 x_2} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_1} u_{tt}\|_{L^2(Q_s)}^2 + \|J_{x_2} u_{tt}\|_{L^2(Q_s)}^2 \right\}. \tag{3.11}
\]

To continue, we introduce the new function $\theta$ defined by

\[
\theta(x,t) = \int_t^T u_{\tau\tau} \, d\tau, \tag{3.12}
\]
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then

\[ u_t(x, t) = \theta(x, s) - \theta(x, t), \quad u_t(x, T) = \theta(x, s). \]  \hspace{1cm} (3.13)

Hence

\[ (1 - 2c_6(T - s)) \left( \| \mathcal{J}_{x_1} \theta(x, s) \|^2_{L^2(\Omega)} + \| \mathcal{J}_{x_2} \theta(x, s) \|^2_{L^2(\Omega)} \right) \]
\[ + \| \mathcal{J}_{x_1 x_2} u_{tt}(x, s) \|^2_{L^2(\Omega)} \leq 2c_6 \left\{ \| \mathcal{J}_{x_1 x_2} u_{tt} \|^2_{L^2(Q_s)} + \| \mathcal{J}_{x_1} \theta \|^2_{L^2(Q_s)} + \| \mathcal{J}_{x_2} \theta \|^2_{L^2(Q_s)} \right\}. \]  \hspace{1cm} (3.14)

Consequently, if \( s_0 > 0 \) satisfies

\[ (1 - 2c_6(T - s)) = \frac{1}{2}, \]  \hspace{1cm} (3.15)

then (3.14) implies

\[ \| \mathcal{J}_{x_1 x_2} u_{tt}(x, s) \|^2_{L^2(\Omega)} + \| \mathcal{J}_{x_1} \theta(x, s) \|^2_{L^2(\Omega)} + \| \mathcal{J}_{x_2} \theta(x, s) \|^2_{L^2(\Omega)} \]
\[ \leq 2c_6 \left\{ \| \mathcal{J}_{x_1 x_2} u_{tt} \|^2_{L^2(Q_s)} + \| \mathcal{J}_{x_1} \theta \|^2_{L^2(Q_s)} + \| \mathcal{J}_{x_2} \theta \|^2_{L^2(Q_s)} \right\}, \]  \hspace{1cm} (3.16)

for all \( s \in [T - s_0, T] \).

If we denote the sum of terms involving norms on the right-hand side of (3.16) by \( y(s) \), we obtain

\[ -\frac{dy(s)}{ds} \leq 4c_6y(s). \]  \hspace{1cm} (3.17)

Integrating (3.17) over \((s, T)\) and taking into account that \( y(T) = 0 \), we get

\[ y(s)e^{4c_6 s} \leq 0. \]  \hspace{1cm} (3.18)

It follows then from (3.18) that \( \mathcal{J}_{x_1 x_2} \omega = 0 \) almost everywhere in \( Q_{T - s_0} \).

To conclude, we prove Theorem 3.1. We should prove the validity of the equality \( R(L) = F \).

Since \( F \) is a Hilbert space, \( R(L) = F \) holds, if

\[ (Lu, W)_F = \int_Q \mathcal{J}_{x_1 x_2} (Lu) \cdot \mathcal{J}_{x_1 x_2} \omega \, dx \, dt + \int_\Omega \mathcal{J}_{x_1} \ell_1 u \cdot \mathcal{J}_{x_1} \omega_0 \, dx \]
\[ + \int_\Omega \mathcal{J}_{x_2} \ell_1 u \cdot \mathcal{J}_{x_2} \omega_0 \, dx + \int_\Omega \mathcal{J}_{x_1 x_2} \ell_2 u \cdot \mathcal{J}_{x_1 x_2} \omega_1 \, dx \]  \hspace{1cm} (3.19)

\[ = 0, \]

it follows that \( \omega = 0 \), \( \omega_0 = 0 \), and \( \omega_1 = 0 \), almost everywhere in \( Q \), where \( W = (\omega, \omega_0, \omega_1) \in R(L) \).

Putting \( u \in D_0(L) \) into (3.19), we obtain
\[
\int_Q \mathcal{J}_{x_1 x_2} (Lu) \cdot \mathcal{J}_{x_1 x_2} \omega \, dx \, dt = 0.
\] (3.20)

Hence, Proposition 3.2 implies that \( \omega = 0 \). Thus (3.19) takes the form
\[
\int_\Omega \mathcal{J}_{x_1} \ell_1 u \cdot \mathcal{J}_{x_1} \omega_0 \, dx + \int_\Omega \mathcal{J}_{x_2} \ell_1 u \cdot \mathcal{J}_{x_2} \omega_0 \, dx
\]
\[
+ \int_\Omega \mathcal{J}_{x_1 x_2} \ell_2 u \cdot \mathcal{J}_{x_1 x_2} \omega_1 \, dx = 0, \quad \forall u \in D_0(L).
\] (3.21)

Since the sets \( \ell_1 u \) and \( \ell_2 u \) are independent and the ranges of the trace operators \( \ell_1 \) and \( \ell_2 \) are everywhere dense in the Hilbert spaces having the norms \( \left( \int_\Omega (\mathcal{J}_{x_1} \omega_0)^2 + (\mathcal{J}_{x_2} \omega_0)^2 \right)^{1/2} \) and \( \left( \int_\Omega (\mathcal{J}_{x_1 x_2} \omega_1)^2 \right)^{1/2} \), respectively, then \( \omega_0 = 0, \omega_1 = 0 \), almost everywhere in \( \Omega \). This completes the proof of Theorem 3.1. \( \square \)

Remark 3.4. The above used method can be easily applied to solve the following differential problem of higher order
\[
Lu = u_{tt} + (-1)^m a(t) \Delta^{2m} u = f(x, t),
\]
\[
\ell_1 u = u(x, 0) = \varphi(x), \quad \ell_2 u = u_t(x, 0) = \beta(x), \quad x \in \Omega,
\]
\[
\int_0^{b_1} x_t^k u(x_1, x_2, t) \, dx_1 \, dx_2 = 0, \quad k = 0, \ldots, 2m-1; \ i = 1, 2,
\]
\[
x = (x_1, x_2) \in \Omega = (0, b_1) \times (0, b_2) \subset \mathbb{R}^2, \quad t \in (0, T).
\] (3.22)

References


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Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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