SOME RESULTS FOR THE APOSTOL-GENOCCHI POLYNOMIALS OF HIGHER ORDER

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Abstract. The present paper deals with multiplication formulas for the Apostol-Genocchi polynomials of higher order and deduces some explicit recursive formulas. Some earlier results of Carlitz and Howard in terms of Genocchi numbers can be deduced. We introduce the 2-variable Apostol-Genocchi polynomials and then we consider the multiplication theorem for 2-variable Genocchi polynomials. Also we introduce generalized Apostol-Genocchi polynomials with a, b, c parameters and we obtain several identities on generalized Apostol-Genocchi polynomials with a, b, c parameters .

Keywords and Phrases:Apostol-Genocchi numbers and polynomials (of higher order), Generalization of Genocchi numbers and polynomials, Raabe's multiplication formula, multiplication formula, Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers

1. Preliminaries and motivation

The classical Genocchi numbers can be defined in a number of ways. The way in which it is defined is often determined by which sorts of applications they are intended to be used for. The Genocchi numbers have wide-ranging applications from number theory and Combinatorics to numerical analysis and other fields of applied mathematics. There exist two important definitions of the Genocchi numbers: the generating function definition, which is the most commonly used definition, and a Pascal-type triangle definition, first given by Philip Ludwig von Seidel, and discussed in [29]. As such, it makes it very appealing for use in combinatorial applications. The idea behind this definition, as in Pascal's triangle, is to utilize a recursive relationship giving some initial conditions to generate the Genocchi numbers. The combinatorics of the Genocchi numbers were developed by Dumont in [4] and various co-authors in the 70s and 80s. Dumont and Foata introduced in 1976

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a three-variable symmetric refinement of Genocchi numbers, which satisfies a simple recurrence relation. A six-variable generalization with many similar properties was later considered by Dumont. In [30] Jang et al. defined a new generalization of Genocchi numbers, poly Genocchi numbers. Kim in [10] gave a new concept for the q-extension of Genocchi numbers and gave some relations between q-Genocchi polynomials and q-Euler numbers. In [31], Simsek et al. investigated the q-Genocchi zeta function and L-function by using generating functions and Mellin transformation. Genocchi numbers are known to count a large variety of combinatorial objects, among which numerous sets of permutations. One of the applications of Genocchi numbers that was investigated by Jeff Remmel in [32] is counting the number of up-down ascent sequences. Another application of Genocchi numbers is in Graph Theory. For instance, Boolean numbers of the associated Ferrers Graphs are the Genocchi numbers of the second kind [33]. A third application of Genocchi numbers is in Automata Theory. One of the generalizations of Genocchi numbers that was first proposed by Han in [34] proves useful in enumerating the class of deterministic finite automata (DFA) that accept a finite language and in enumerating a generalization of permutations counted by Dumont. Recently S. Herrmann in [6], presented a relation between the f -vector of the boundary and the interior of a simplicial ball directly in terms of the f-vectors. The most interesting point about this equation is the occurrence of the Genocchi numbers G_{2n}. In the last decade, a surprising number of papers appeared proposing new generalizations of the classical Genocchi polynomials to real and complex variables or treating other topics related to Genocchi polynomials. Qiu-Ming Luo in [19] introduced new generalizations of Genocchi polynomials, he defined the Apostol-Genocchi polynomials of higher order and q-Apostol-Genocchi polynomials and he obtained a relationship between Apostol-Genocchi polynomials of higher order and Goyal-Laddha-Hurwitz-Lerch Zeta function. Next Qiu-Ming Luo and H.M. Srivastava in [35] by Apostol-Genocchi polynomials of higher order derived various explicit series representations in terms of the Gaussian hypergeometric function and the Hurwitz (or generalized) zeta function which yields a deeper insight into the effectiveness of this type of generalization. Also it is clear that Apostol-Genocchi polynomials of higher order are in a class of orthogonal polynomials and we know that most such special functions that are orthogonal are satisfied in multiplication theorem, so in this present paper we show this property is true for Apostol-Genocchi polynomials of higher order.

The study of Genocchi numbers and their combinatorial relations has received much attention [2, 4, 6, 10, 13, 19, 22, 23, 26, 27, 29, 37]. In this paper we consider some combinatorial relationships of the Apostol-Genocchi numbers of higher order.

The unsigned Genocchi numbers $\{G_{2n}\}_{n\geqslant 1}$ can be defined through their generating function:

$$\sum_{n=1}^{\infty} G_{2n} \frac{x^{2n}}{(2n)!} = x.\tan\left(\frac{x}{2}\right)$$

and also

$$\sum_{n\geqslant 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!} = -t \tanh\left(\frac{t}{2}\right)$$

So, by simple computation

$$\begin{aligned}
\tanh\left(\frac{t}{2}\right) &= \sum_{s\geqslant 0} \frac{(\frac{t}{2})^{2s+1}}{(2s+1)!} . \sum_{m\geqslant 0} (-1)^m E_{2m} \frac{(\frac{t}{2})^{2m}}{(2m)!} \\
&= \sum_{s,m\geqslant 0} \frac{(-1)^m}{2^{2m+2s+1}} \frac{E_{2m} t^{2m+2s+1}}{(2m)!(2s+1)!} \\
&= \sum_{n\geqslant 1} \sum_{m=0}^{n-1} \binom{2n-1}{2m} \frac{(-1)^m E_{2m} t^{2n-1}}{2^{2n-1}(2n-1)!},
\end{aligned}$$

we obtain for $n \geqslant 1$,

$$G_{2n} = \sum_{k=0}^{n-1} (-1)^{n-k-1} (n-k) \binom{2n}{2k} \frac{E_{2k}}{2^{2n-2}}$$

where E_k are Euler numbers. Also the Genocchi numbers G_n are defined by the generating function

$$G(t) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, (|t| < \pi).$$

In general, it satisfies $G_0 = 0, G_1 = 1, G_3 = G_5 = G_7 = ...G_{2n+1} = 0$, and even coefficients are given $G_{2n} = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}$, where B_n are Bernoulli numbers and E_n are Euler numbers. The first few Genocchi numbers for even integers are -1,1,-3,17,-155,2073,... . The first few prime Genocchi numbers are -3 and 17, which occur at $n = 6$ and 8. There are no others with $n < 10^5$. For $x \in \mathbb{R}$, we consider the Genocchi polynomials as follows

$$G(x,t) = G(t)e^{xt} = \frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.$$

In special case $x = 0$, we define $G_n(0) = G_n$. Because we have

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k},$$

It is easy to deduce that $G_k(x)$ are polynomials of degree k. Here, we present some of the first Genocchi's polynomials:

$$G_1(x) = 1, G_2(x) = 2x-1, G_3(x) = 3x^2-3x, G_4(x) = 4x^3-6x^2+1,$$

$$G_5(x) = 5x^4 - 10x^3 + 5x, G_6(x) = 6x^5 - 15x^4 + 15x^2 - 3, \ ...$$

The classical Bernoulli polynomials (of higher order) $B_n^{(\alpha)}(x)$ and Euler polynomials (of higher order) $E_n^{(\alpha)}(x), (\alpha \in \mathbb{C})$, are usually defined by means of the following generating functions [11, 13, 15, 21, 24, 25, 28]

$$\left(\frac{z}{e^z - 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!}, (|z| < 2\pi)$$

and

$$\left(\frac{2}{e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!}, (|z| < \pi)$$

So that, obviously,

$$B_n(x) := B_n^1(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x).$$

In 2002, Q. M. Luo et al. (see [5, 17, 18]) defined the generalization of Bernoulli polynomials and Euler numbers, as follows

$$\frac{tc^{xt}}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(x; a, b, c)}{n!} t^n, (|t \ln \frac{b}{a}| < 2\pi)$$

$$\frac{2}{b^t + a^t} = \sum_{n=0}^{\infty} E_n(a, b)\frac{t^n}{n!}, (|t \ln \frac{b}{a}| < \pi).$$

Here, we give an analogous definition for generalized Apostol-Genocchi polynomials.

Let $a, b > 0$, The Generalized Apostol-Genocchi Numbers and Apostol-Genocchi polynomials with a, b, c parameters are defined by

$$\frac{2t}{\lambda b^t + a^t} = \sum_{n=0}^{\infty} G_n(a, b; \lambda)\frac{t^n}{n!}$$

$$\frac{2t}{\lambda b^t + a^t} e^{xt} = \sum_{n=0}^{\infty} G_n(x, a, b; \lambda)\frac{t^n}{n!}$$

$$\frac{2t}{\lambda b^t + a^t} c^{xt} = \sum_{n=0}^{\infty} G_n(x, a, b, c; \lambda)\frac{t^n}{n!}$$

respectively.

For a real or complex parameter α, The Apostol-Genocchi polynomials with a, b, c parameters of order α, $G_n^{(\alpha)}(x; a, b; \lambda)$, each of degree n is x as well as in α, are defined by the following generating functions

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, a, b; \lambda)\frac{t^n}{n!},$$

Clearly, we have $G_n^{(1)}(x, a, b; \lambda) = G_n(x; a, b; \lambda)$.

Now, we introduce the 2-variable Apostol-Genocchi polynomials and then we consider the multiplication theorem for 2-variable Apostol-Genocchi Polynomials.

We start with the definition of Apostol-Genocchi polynomials $G_n(x;\lambda)$. The Apostol-Genocchi Polynomials $G_n(x;\lambda)$ in variable x are defined by means of the generating function

$$\frac{2ze^{xz}}{\lambda e^z+1}=\sum_{n=0}^{\infty}G_n(x;\lambda)\frac{z^n}{n!}\quad (|z|<2\pi \text{ when } \lambda=1, |z|<|\log\lambda| \text{ when } \lambda\neq 1),$$

with, of course,

$$G_n(\lambda):=G_n(0;\lambda),$$

Where $G_n(\lambda)$ denotes the so-called Apostol-Genocchi numbers.

Also (see [1, 14, 16, 19, 20, 24, 28]) Apostol-Genocchi Polynomials $G_n^{(\alpha)}(x;\lambda)$ of order α in variable x are defined by means of the generating function:

$$\left(\frac{2z}{\lambda e^z+1}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}$$

with, of course, $G_n^{(\alpha)}(\lambda):=G_n^{\alpha}(0;\lambda)$.

Where $G_n^{\alpha}(\lambda)$ denotes the so-called Apostol-Genocchi numbers of higher order. If we set,

$$\phi(x,t;\alpha)=\left(\frac{2t}{e^t+1}\right)^{\alpha}e^{xt},$$

then,

$$\frac{\partial\phi}{\partial x}=t\phi,$$

and,

$$t\frac{\partial\phi}{\partial t}-\left\{\frac{\alpha+tx}{t}-\frac{\alpha e^t}{e^t+1}\right\}\frac{\partial\phi}{\partial x}=0.$$

Next, we introduce the class of Apostol-Genocchi numbers as follows. (for more information see [38])

$${}_HG_n(\lambda)=\sum_{s=0}^{[\frac{n}{2}]}\frac{n!G_{n-2s}(\lambda)G_s(\lambda)}{s!(n-2s)!}$$

The generating function of ${}_HG_n(\lambda)$ is provided by

$$\frac{4t^3}{(\lambda e^t+1)(\lambda e^{t^2}+1)}=\sum_{n=0}^{\infty}\ {}_HG_n(\lambda)\frac{t^n}{n!}$$

and the generalization of ${}_HG_n(\lambda)$ for $(a,b)\neq 0$, is

$$\frac{4t^3}{(\lambda e^{at}+1)(\lambda e^{bt^2}+1)}=\sum_{n=0}^{\infty}\ {}_HG_n(a,b;\lambda)\frac{t^n}{n!}$$

where

$$_{H}G_n(a,b;\lambda)=\frac{1}{ab}\sum_{n=0}^{[\frac{n}{2}]}\frac{n!a^{n-2s}b^sG_{n-2s}(\lambda)G_s(\lambda)}{s!(n-2s)!}$$

The main object of the present paper is to investigate the multiplication formulas for the Apostol-type polynomials.

Luo in [16] defined the multiple alternating sums as

$$Z_k^{(l)}(m;\lambda)=(-1)^l\sum_{\substack{0\leq v_1,v_2,\ldots,v_m\leq l\\ v_1+v_2+\ldots+v_m=\ell}}\binom{l}{v_1,v_2,\ldots,v_m}(-\lambda)^{v_1+2v_2+\ldots+mv_m}$$

$$Z_k(m;\lambda)=\sum_{j=1}^{m}(-1)^{j+1}\lambda^j j^k=\lambda-\lambda^2 2^k+\ldots+(-1)^{m+1}\lambda^m m^k$$

$$Z_k(m)=\sum_{j=1}^{m}(-1)^{j+1}j^k=1-2^k+\ldots+(-1)^{m+1}m^k,\ (m,k,l\in\mathbb{N}_0;\lambda\in\mathbb{C})$$

where $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $(\mathbb{N}:=\{1,2,3,\ldots\})$.

2. THE MULTIPLICATION FORMULAS FOR THE APOSTOL-GENOCCHI POLYNOMIALS OF HIGHER ORDER

In this Section, we obtain some interesting new relations and properties associated with Apostol-Genocchi polynomials of higher order and then derive several elementary properties including recurrence relations for Genocchi numbers. First of all we prove the multiplication theorem of these polynomials.

Theorem 2.1. *For $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha,\lambda\in\mathbb{C}$, the following multiplication formula of the Apostol-Genocchi polynomials of higher order holds true:*
(1)

$$G_n^{(\alpha)}(mx;\lambda)=m^{n-\alpha}\sum_{v_1,v_2,\ldots,v_{m-1}\geq 0}\binom{\alpha}{v_1,v_2,\ldots,v_{m-1}}(-\lambda)^r G_n^{(\alpha)}\left(x+\frac{r}{m};\lambda^m\right)$$

where $r=v_1+2v_2+\ldots+(m-1)v_{m-1}$, ($m$ is odd)

Proof. It is easy to observe that

$$\frac{1}{\lambda e^t+1}=-\frac{1-\lambda e^t+\lambda^2e^{2t}+\ldots+(-\lambda)^{m-1}e^{(m-1)t}}{(-\lambda)^me^{mt}-1}\qquad(*)$$

But we have, if $x_i\in\mathbb{C}$

$$(x_1+x_2+\ldots+x_m)^n=\sum_{\substack{a_1,a_2,\ldots,a_m\geqslant 0\\ a_1+a_2+\ldots a_m=n}}\binom{n}{a_1,a_2,\ldots,a_m}x_1^{a_1}x_2^{a_2}\ldots x_m^{a_m}\qquad(**)$$

The last summation takes place over all positive or zero integers $a_i \geqslant 0$ such that $a_1 + a_2 + ... + a_m = n$, where

$$\binom{n}{a_1, a_2, ..., a_m} := \frac{n!}{a_1! a_2! ... a_m!}$$

So by applying $(*)$ on the following first equality sign and setting $(x_1 = 1, x_k = (-\lambda)^k e^{kt} \quad \text{for} \quad k \geq 2)$ and $n = \alpha$ in $(**)$ on the following second equality sign, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(\alpha)}(mx;\lambda)\frac{t^n}{n!} &= \Big(\frac{2t}{\lambda e^t + 1}\Big)^{\alpha} e^{mxt} \\
&= \Big(\frac{2t}{\lambda^m e^{mt} + 1}\Big)^{\alpha} \Big(\sum_{k=0}^{m-1}(-\lambda)^k e^{kt}\Big)^{\alpha} e^{mxt} \\
&= \sum_{v_1,v_2,...,v_{m-1} \geqslant 0} \binom{\alpha}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r \Big(\frac{2t}{\lambda^m e^{mt} + 1}\Big)^{\alpha} e^{(x+\frac{r}{m})mt} \\
&= \sum_{n=0}^{\infty} \Big(m^{n-\alpha} \sum_{v_1,v_2,...,v_m \geqslant 0} \binom{\alpha}{v_1, v_2, ..., v_m} (-\lambda)^r G_n^{(\alpha)}\Big(x + \frac{r}{m}; \lambda^m\Big)\Big)\frac{t^n}{n!}
\end{aligned}$$

By comparing the coefficient of $\frac{t^n}{n!}$ on both sides of last equation, proof is complete. □

In terms of the generalized Apostol-Genocchi polynomials, by setting $\lambda = 1$ in Theorem 2.1, we obtain the following explicit formula that is called multiplication theorem for Genocchi polynomials of higher order.

Corollary 2.2. *For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\alpha, \in \mathbb{C}$, we have*

$$G_n^{(\alpha)}(mx) = m^{n-\alpha} \sum_{v_1,v_2,...,v_{m-1} \geqslant 0} \binom{\alpha}{v_1, v_2, ..., v_{m-1}} (-1)^r G_n^{(\alpha)}\Big(x + \frac{r}{m}\Big) \quad (m \text{ is odd}).$$

And using Corollary 2.2, (by setting $\alpha = 1$), we get Corollary 2.3 that is the main result of [36] and is called multiplication Theorem for Genocchi polynomials.

Corollary 2.3. *For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, we have*

$$G_n(mx) = m^{n-1} \sum_{k=0}^{m-1} (-1)^k G_n\Big(x + \frac{k}{m}\Big) \quad (m \text{ is odd}).$$

Now, we consider the multiplication formula for the Apostol-Genocchi numbers when m is even.

Theorem 2.4. *For $m \in \mathbb{N}$ (m even), $n \in \mathbb{N}$, $\alpha, \lambda \in \mathbb{C}$, the following multiplication formula of the Apostol-Genocchi polynomials of higher order* *holds true:*

$$G_n^{(\alpha)}(mx;\lambda)=(-2)^{\alpha}m^{n-\alpha}\sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0}\binom{\alpha}{v_1,v_2,\ldots,v_{m-1}}(-\lambda)^r B_n^{(\alpha)}\Big(x+\frac{r}{m},\lambda^m\Big),$$

where $r=v_1+2v_2+\ldots+(m-1)v_{m-1}$.

Proof. It is easy to observe that

$$\frac{1}{\lambda e^t+1}=-\frac{1-\lambda e^t+\lambda^2e^{2t}+\ldots+(-\lambda)^{m-1}e^{(m-1)t}}{(-\lambda)^m e^{mt}-1}$$

So, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}G_n^{(\alpha)}(mx;\lambda)\frac{t^n}{n!} &= \Big(\frac{2t}{\lambda e^t+1}\Big)^{\alpha}e^{mxt}\\
&= 2^{\alpha}\Big(\frac{t}{\lambda e^t+1}\Big)^{\alpha}e^{mxt}\\
&= (-2)^{\alpha}\Big(\frac{t}{\lambda^m e^{mt}-1}\Big)^{\alpha}\Big(\sum_{k=0}^{m-1}(-\lambda e^t)^k\Big)^{\alpha}e^{mxt}\\
&= (-2)^{\alpha}\sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0}\binom{\alpha}{v_1,v_2,\ldots,v_{m-1}}(-\lambda)^r\Big(\frac{t}{\lambda^m e^m-1}\Big)^{\alpha}e^{(x+\frac{r}{m})mt}\\
&= \sum_{n=0}^{\infty}\Big((-2)^{\alpha}m^{n-\alpha}\sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0}\binom{\alpha}{v_1,v_2,\ldots,v_{m-1}}(-\lambda)^r\\
&\times\ B_n^{(\alpha)}(x+\frac{r}{m};\lambda^m)\Big)\frac{t^n}{n!}
\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides proof will be complete. □

Next, using Theorem 2.4, (with $\lambda=1$), we obtain the Genocchi polynomials of higher order can be expressed by the Bernoulli polynomials of higher order when m is even

Corollary 2.5. *For* $m\in\mathbb{N}$ *(*m *even),* $n\in\mathbb{N}_0$, $\alpha\in\mathbb{C}$, *we get*

$$G_n^{(\alpha)}(mx)=(-2)^{\alpha}m^{n-\alpha}\sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0}\binom{\alpha}{v_1,v_2,\ldots,v_{m-1}}(-1)^r B_n^{\alpha}\Big(x+\frac{r}{m}\Big).$$

Also by applying $\alpha=1$, in corollary 2.5 we obtain the following assertion that is one of the most remarkable identities in area of Genocchi polynomials.

Corollary 2.6. *For* $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, *we obtain*

$$G_n(mx)=-2m^{n-1}\sum_{k=0}^{m-1}(-1)^k B_n\Big(x+\frac{k}{m}\Big)\quad m\ \textit{is even.}$$

Obviously, the result of Corollary 2.6 is analogous with the well-known Raabe's multiplication formula. Now, we present explicit evaluations of $Z_n^{(l)}(m;\lambda)$, $Z_n^{(l)}(\lambda)$, $Z_n(m)$ by Apostol-Genocchi polynomials.

Theorem 2.7. *For $m, n, l \in \mathbb{N}_0, \lambda \in \mathbb{C}$, we have*

$$Z_n^{(l)}(m;\lambda) = 2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{j(m+1)}\lambda^{mj+l}}{(n+1)_l}\sum_{k=0}^{n+l}\binom{n+l}{k}G_k^{(j)}(mj+l;\lambda)G_{n+l-k}^{(l-j)}(\lambda)$$

where $(n)_0 = 1, (n)_k = n(n+1)...(n+k-1)$.

Proof. By definition of $Z_n^{(l)}(m;\lambda)$, we calculate the following sum

$$\sum_{n=0}^{\infty} Z_n^{(l)}(m;\lambda)\frac{t^n}{n!} =$$

$$\sum_{n=0}^{\infty}\Bigg[(-1)^l\sum_{\substack{0\leqslant v_1,v_2,...,v_m\leqslant l\\ v_1+v_2+...+v_m=l}}\binom{l}{v_1,v_2,...,v_m}(-\lambda)^{\lambda_1+2\lambda_2+...+m\lambda_m}(v_1+2v_2+...+mv_m)^n\Bigg]\frac{t^n}{n!}$$

$$\begin{aligned}
&= (-1)^l\sum_{\substack{0\leqslant v_1,v_2,...,v_m\leqslant l\\ v_1+v_2+...+v_m=l}}\binom{l}{v_1,v_2,...,v_m}(-\lambda e^t)^{\lambda_1+2\lambda_2+...+m\lambda_m}\\
&= (\lambda e^t-\lambda^2e^{2t}+...+(-1)^{m+1}\lambda^m e^{mt})^l\\
&= \left(\frac{(-1)^{m+1}\lambda^{m+1}e^{(m+1)t}}{\lambda e^t+1}+\frac{\lambda e^t}{\lambda e^t+1}\right)^l\\
&= (2t)^{-l}\sum_{j=0}^{l}\binom{l}{j}\Big[\frac{2t(-1)^{m+1}\lambda^{m+1}e^{(m+1)t}}{\lambda e^t+1}\Big]^j\Big[\frac{2t\lambda e^t}{\lambda e^t+1}\Big]^{l-j}\\
&= (2t)^{-l}\sum_{j=0}^{l}\binom{l}{j}(-1)^{j(m+1)}\lambda^{mj+l}\sum_{n=0}^{\infty}G_n^{(j)}(mj+l;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}G_n^{(l-j)}(\lambda)\frac{t^n}{n!}\\
&= 2^{-l}\sum_{n=0}^{\infty}\Bigg[\sum_{j=0}^{l}\binom{j}{l})\frac{(-1)^{j(m+1)}\lambda^{mj+l}}{(n+1)_l}\sum_{k=0}^{n+l}\binom{n+l}{k}G_k^{(j)}(mj+l;\lambda)G_{n+l-k}^{(l-j)}(\lambda)\Bigg]\frac{t^n}{n!}
\end{aligned}$$

by comparing the coefficients of $\frac{t^n}{n!}$ on both sides, proof will be complete. □

As a direct result, using $\lambda = 1$ in Theorem 2.7, we derive an explicit representation of multiple alternating sums $Z_n^{(l)}(m)$, in terms of the Genocchi polynomials of higher order. We also deduce their special cases and applications which lead to the corresponding results for the Genocchi polynomials.

Corollary 2.8. *For $m, n, l \in \mathbb{N}_0$, the following formula holds true in terms of the Genocchi polynimials*

$$Z_n^{(l)}(m) = 2^{-l}\sum_{j=0}^{l}\binom{l}{j}\frac{(-1)^{j(m+1)}}{(n+1)_l}\sum_{k=0}^{n+l}\binom{n+l}{k}G_k^{(j)}(mj+l)G_{n+l-k}^{l-j}$$

where $(n)_0 = 1, (n)_k = n(n+1)...(n+k-1)$.

Next we investigate some of the recursive formulas for the Apostol-Genocchi numbers of higher order that are analogous to the results of Howard [7, 8, 9] and we deduce that they constitute a useful special case.

Theorem 2.9. *Let m be odd, $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, then we have*

$$m^n G_n^{(l)}(\lambda^m) - m^l G_n^{(l)}(\lambda) = (-1)^{l-1} \sum_{k=0}^{n} \binom{n}{k} m^k G_k^{(l)}(\lambda^m) Z_{n-k}^{(l)}(m-1;\lambda).$$

Proof. By taking $x = 0, \alpha = l$ in (1), where $r = v_1 + 2v_2 + ... + (m-1)v_{m-1}$ we obtain

$$m^l G_n^{(l)}(\lambda) = m^n \sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r G_n^{(l)}(\frac{r}{m}, \lambda^m)$$

But we know

$$G_n^{(l)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(l)}(\lambda) x^{n-k}$$

So, we obtain

$$\begin{aligned} m^l G_n^{(l)}(\lambda) &= m^n \sum_{v_1,v_2,\ldots,v_{m-1}\geqslant 0} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r \sum_{k=0}^{n} \binom{n}{k} G_k^{(l)}(\lambda^m) \Big(\frac{r}{m}\Big)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} m^k G_k^{(l)}(\lambda^m) \sum_{0\leqslant v_1,v_2,\ldots,v_{m-1}\leqslant l} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r r^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} m^k G_k^{(l)}(\lambda^m) \sum_{\substack{0\leqslant v_1,v_2,\ldots,v_{m-1}\leqslant l \\ v_1+v_2+\ldots v_{m-1}=l}} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r r^{n-k} + m^n G_n^{(l)}(\lambda^m) \\ &= (-1)^l \sum_{k=0}^{n} \binom{n}{k} m^k G_k^{(l)}(\lambda^m) Z_{n-k}^{(l)}(m-1;\lambda) + m^n G_n^{(l)}(\lambda^m) \end{aligned}$$

So proof is complete. □

Furthermore, we derive some well-known results (see [10]) involving Genocchi polynomials of higher order and Genocchi polynomials which we state here. By setting $\lambda = 1$, $l = 1$ in Theorem 2.9, we get Corollaries 2.10, 2.11, respectively.

Corollary 2.10. *Let m be odd, $n, l \in \mathbb{N}_0$, then we have*

$$(m^n - m^l) G_n^{(l)} = (-1)^{l-1} \sum_{k=0}^{n} \binom{n}{k} G_k^{(l)} Z_{n-k}^{(l)}(m-1).$$

Corollary 2.11. *Let m be odd, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, then we have*

$$m^n G_n(\lambda^m) - m G_n(\lambda) = \sum_{k=0}^{n} \binom{n}{k} m^k G_k(\lambda^m) Z_{n-k}(m-1;\lambda).$$

Also by setting $\lambda = 1$ in Corollary 2.11, we get the following assertion that is analogous to the formula of Howard in terms of Genocchi numbers.

Corollary 2.12. *For m be odd, $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we obtain*

$$(m^n - m)G_n = \sum_{k=0}^{n} \binom{n}{k} m^k G_k Z_{n-k}(m-1)$$

.

Next, we investigate the generalization of Howard's formula in terms of Apostol-Genocchi numbers, when m is even.

Theorem 2.13. *Let m be even, $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, the following formula*

$$m^l G_n^{(l)}(\lambda) - (-2)^l m^n B_n^{(l)}(\lambda^m) = 2^l \sum_{k=0}^{n} \binom{n}{k} m^k B_k^{(l)}(\lambda^m) Z_{n-k}^{(l)}(m-1;\lambda)$$

holds true, where $r = v_1 + 2v_2 + ... + (m-1)v_{m-1}$.

Proof. We have

$$G_n^{(l)}(\lambda) = (-2)^l m^{n-l} \sum_{v_1,v_2,...,v_{m-1}\geqslant 0} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r B_n^{(l)}(\frac{r}{m}, \lambda^m)$$

But we know

$$B_n^{(l)}(x;\lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(l)}(\lambda) x^{n-k}$$

So we get

$$\begin{aligned} m^l G_n^{(l)}(\lambda) &= (-2)^l m^n \sum_{v_1,v_2,...,v_{m-1}\geqslant 0} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r \sum_{k=0}^{n} \binom{n}{k} B_k^{(l)}(\lambda^m) \left(\frac{r}{m}\right)^{n-k} \\ &= (-2)^l \sum_{k=0}^{n} \binom{n}{k} m^k B_k^{(l)}(\lambda^m) \sum_{v_1,v_2,...,v_{m-1}\geqslant 0} \binom{l}{v_1, v_2, ..., v_{m-1}} (-\lambda)^r r^{n-k} \\ &= 2^l \sum_{k=0}^{n} \binom{n}{k} m^k B_k^{(l)}(\lambda^m) Z_{n-k}^{(l)}(m-1;\lambda) + (-2)^l m^n B_n^{(l)}(\lambda^m) \end{aligned}$$

So we obtain

$$m^l G_n^{(l)}(\lambda) - (-2)^l m^n B_n^{(l)}(\lambda^m) = 2^l \sum_{k=0}^{n} \binom{n}{k} m^k B_k^{(l)}(\lambda^m) Z_{n-k}^{(l)}(m-1;\lambda)$$

So the proof is complete. □

Also by letting $\lambda = 1$ in Theorem 2.13, we obtain the following assertion.

Corollary 2.14. *Let m be even, $n, l \in \mathbb{N}_0$,then we get*

$$m^l G_n^{(l)} - (-2)^l m^n B_n^{(l)} = 2^l \sum_{k=0}^{n} \binom{n}{k} m^k B_n^{(l)} Z_{n-k}^{(l)}(m-1)$$

Here we present a recurrence relation for Apostol-Genocchi numbers of higher order.

Theorem 2.15. *Let $n, k \geqslant 1$, then we have*

$$G_k^{(n+1)}(\lambda) = 2kG_{k-1}^{(n)}(\lambda) - \Big(2 - \frac{2k}{n}\Big)G_k^{(n)}(\lambda)$$

Proof. Let us put $G_n(t;\lambda) = \Big(\frac{2t}{\lambda e^t+1}\Big)^n$. Then $G_n(t;\lambda)$ is the generating function of higher order Apostol-Genocchi numbers. The derivative $G^{'}(t;\lambda) = \frac{d}{dt}G_n(t;\lambda)$ is equal to

$$n\Big(\frac{1}{t} - \frac{\lambda e^t}{\lambda e^t + 1}\Big)G_n(t;\lambda) = \frac{n}{t}G_n(t;\lambda) - nG_n(t;\lambda) + \frac{n}{\lambda e^t + 1}G_n(t;\lambda)$$

and

$$tG_n^{'}(t;\lambda) = nG_n(t;\lambda) - ntG_n(t;\lambda) + \frac{n}{2}G_{n+1}(t)$$

so we obtain

$$\frac{G_k^{(n)}(\lambda)}{(k-1)!} = n\frac{G_k^{(n)}(\lambda)}{k!} - n\frac{G_{k-1}^{(n)}(\lambda)}{(k-1)!} + \frac{n}{2}\frac{G_k^{(n+1)}(\lambda)}{k!}$$

for $k \geqslant 1$. This formula can written as

$$G_k^{(n+1)}(\lambda) = 2kG_{k-1}^{(n)}(\lambda) - \Big(2 - \frac{2k}{n}\Big)G_k^{(n)}(\lambda)$$

so proof is complete. □

3. GENERALIZED APOSTOL GENOCCHI POLYNOMIALS WITH a, b, c PARAMETERS

In this section we investigate some recurrence formulas for generalized Apostol-Genocchi polynomials with a, b, c parameters . In 2003, Cheon [3] rederived several known properties and relations involving the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ by making use of some standard techniques based upon series rearrangement as well as matrix representation. Srivastava and Pinter [36] followed Cheon's work [3] and established two relations involving the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$. So, we will study further the relations between generalized Bernoulli polynomials with a, b parameters and Genocchi polynomials with the methods of generating function and series rearrangement.

Theorem 3.1. *Let $x \in \mathbb{R}$ and $n \geqslant 0$. For every positive real number a, b and c such that $a \neq b$ and $b > 0$, we have*

$$G_n^{(\alpha)}(a, b; \lambda) = G_n^{(\alpha)}\Big(\frac{\alpha \ln a}{\ln a - \ln b}; \lambda\Big)(\ln b - \ln a)^{n-\alpha}$$

Proof. We know

$$\begin{aligned}
\Big(\frac{2t}{\lambda b^t + a^t}\Big)^{\alpha} &= \sum_{n=0}^{\infty} G_n^{(\alpha)}(a,b;\lambda)\frac{t^n}{n!} \\
&= \frac{1}{a^{\alpha t}}\Big(\frac{2t}{\lambda e^{t(\ln b - \ln a)}+1}\Big)^{\alpha} \\
&= e^{-t\alpha \ln a}\Big(\frac{2t(\ln b - \ln a)}{\lambda e^{t(\ln b - \ln a)}+1}\Big)^{\alpha} \times \frac{1}{(\ln b - \ln a)^{\alpha}} \\
&= \frac{1}{(\ln b - \ln a)^{\alpha}}\sum_{n=0}^{\infty} G_n^{(\alpha)}\Big(\frac{\alpha \ln a}{\ln a - \ln b};\lambda\Big)(\ln b - \ln a)^n \frac{t^n}{n!}
\end{aligned}$$

So by comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we get

$$G_n^{(\alpha)}(a,b;\lambda) = G_n^{(\alpha)}\Big(\frac{\alpha \ln a}{\ln a - \ln b};\lambda\Big)(\ln b - \ln a)^{n-\alpha}.$$

□

Theorem 3.2. *Let $x \in \mathbb{R}$ and $n \geqslant 0$. For every positive real number a, b and c such that $a \neq b$ and $b > 0$, we have*

$$G_n^{(\alpha)}(x;a,b,c;\lambda) = G_n^{(\alpha)}\Big(\frac{-\alpha \ln a + x \ln c}{\ln b - \ln a},\lambda\Big)(\ln b - \ln a)^{n-\alpha}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;a,b,c;\lambda) &= \Big(\frac{2t}{\lambda b^t + a^t}\Big)^{\alpha} c^{xt} \\
&= \frac{1}{\alpha^{at}}\Big(\frac{2t}{\lambda e^{t(\ln b - \ln a)}+1}\Big)^{\alpha} c^{xt} \\
&= e^{t(-\alpha \ln a + x \ln c)}\Big(\frac{2t}{\lambda e^{t(\ln b - \ln a)}+1}\Big)^{\alpha} \\
&= \frac{1}{(\ln b - \ln a)^{\alpha}}\sum_{n=0}^{\infty} G_n^{(\alpha)}\Big(\frac{-\alpha \ln a + x \ln c}{\ln b - \ln a},\lambda\Big)(\ln b - \ln a)^n \frac{t^n}{n!}.
\end{aligned}$$

So by comparing the coefficient of $\frac{t^n}{n!}$ on both sides, we get

$$G_n^{(\alpha)}(x;a,b,c;\lambda) = G_n^{(\alpha)}\Big(\frac{-\alpha \ln a + x \ln c}{\ln b - \ln a},\lambda\Big)(\ln b - \ln a)^{n-\alpha}$$

Therefore proof is complete. □

The generalized Apostal-Genocchi polynomials of higher order $G_n^{(\alpha)}(x;a,b,c;\lambda)$ possess a number of interesting properties which we state here.

Theorem 3.3. *Let $a, b, c \in \mathbb{R}^+$ $(a \neq b)$ and $x \in \mathbb{R}$, then*

$$G_n^{(\alpha)}(x+1;a,b,c;\lambda) = \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k} G_k^{(\alpha)}(x;a,b,c;\lambda) \tag{2}$$

$$G_n^{(\alpha)}(x+\alpha;a,b,c;\lambda)=G_n^{(\alpha)}\Big(x;\frac{a}{c},\frac{b}{c},c;\lambda\Big) \tag{3}$$

$$G_n^{(\alpha)}(\alpha-x;a,b,c;\lambda)=G_n^{(\alpha)}\Big(-x;\frac{a}{c},\frac{b}{c},c;\lambda\Big) \tag{4}$$

$$G_n^{(\alpha+\beta)}(x+y;a,b,c;\lambda)=\sum_{r=0}^{k}\binom{k}{r}G_{k-r}^{(\alpha)}(x;a,b,c;\lambda)G_r^{(\beta)}(y;a,b,c;\lambda) \tag{5}$$

$$\frac{\partial^l}{\partial x^l}\{G_n^{(\alpha)}(x;a,b,c;\lambda)\}=\frac{n!}{(n-\ell)!}(\ln c)^\ell G_{n-\ell}^{(\alpha)}(x;a,b,c;\lambda) \tag{6}$$

(7)
$$\int_s^t G_n^{(\alpha)}(x;a,b,c;\lambda)dx=\frac{1}{(n+1)\ln c}\Big[G_{n+1}^{(\alpha)}(t;a,b,c;\lambda)-G_{n+1}^{(\alpha)}(s;a,b,c;\lambda)\Big]$$

Proof. We know

$$\begin{aligned}\sum_{n=0}^{\infty}G_n^{(\alpha)}(x+1;a,b,c;\lambda)\frac{t^n}{n!} &= \Big(\frac{t}{\lambda b^t+a^t}\Big)^\alpha .c^{(x+1)t}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}G_k^{(\alpha)}(x;a,b,c;\lambda)(\ln c)^n\frac{t^{n+k}}{n!k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}G_k^{(\alpha)}(x;a,b,c;\lambda)(\ln c)^{n-k}\frac{t^{n+k}}{(n-k)!k!}\end{aligned}$$

So comparing the coefficients of t^n on both sides, we arrive at the result (2) asserted by Theorem 3.3. Similary, by simple manipulations, leads us to the result (3), (4) and (5) of Theorem 3.3 and by successive differentiation with respect to x and then using the principle of mathematical induction on $\ell\in\mathbb{N}_0$, we obtain the formula (6). Also, by taking $\ell=1$ in (6) and integrating both sides with respect to x, we get the formula (7). □

Remark 3.4. *Let $a,b,c\in\mathbb{R}^+$ $(a\neq -b)$ and $x\in\mathbb{R}$, by differentiating both sides of the following generating function*

$$\sum_{n=0}^{\infty}G_n^{\alpha}(x;a,b,c;\lambda)\frac{t^n}{n!}=\frac{t^\alpha}{(\lambda e^{t\ln(\frac{b}{a})}+1)^\alpha}e^{t(x\ln c-x\ln a)},$$

We get,

$$\begin{aligned}\alpha\lambda\ln(\frac{b}{a})\sum_{k=0}^{n}\binom{n}{k}(\ln b)^k G_{n-k}^{(\alpha+1)}(x;a,b,c;\lambda) &= (\alpha-n)G_n^{(\alpha)}(x;a,b,c;\lambda)\\ &+ n(x\ln c-\alpha\ln a)G_{n-1}^{(\alpha)}(x;a,b,c;\lambda).\end{aligned}$$

Remark 3.5. *GI-Sang Cheon and H. M. Srivastava in* [3, 20] *investigated the classical relationship between Bernoulli and Euler polynomials as follows*

$$B_n(x) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x)$$

by applying a similar Srivastava's method in [20] *we obtain the following result for generalized Bernoulli polynomials and Genocchi numbers*

$$\begin{aligned} B_n(x+y,a,b) &= \frac{1}{2}\sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k}[B_k(y,a,b)+B_k(y+1,a,b)]G_{n-k}(x), \\ G_n(x+y) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}[G_k(y)+G_k(y+1)]E_{n-k}(x), \end{aligned}$$

so, because we have

$$G_n(y+1)+G_n(y) = 2ny^{n-1},$$

we obtain

$$G_n(x+y) = \sum_{k=0}^{n} k\binom{n}{k} y^{k-1} E_{n-k}(x) \qquad (y \neq 0).$$

4. Multiplication theorem for 2-variable Genocchi polynomial

We apply the method of generating function, which are exploited to derive further classes of partial sums involving generalized many index many variable polynomials. In introduction we introduced 2-variable Genocchi polynomial. An application of 2-variable Genocchi polynomials is relevant to the multiplication theorems. In this section we develop the multiplication theorem for 2-variable Genocchi polynomial which yields a deeper insight into the effectiveness of this type of generalizations.

Theorem 4.1. *Let $x, y \in \mathbb{R}^+$ and m be odd, we obtain*

$$G_n(mx,py,\lambda) = m^{n-1}\sum_{k=0}^{m-1} \lambda^k(-1)^k_H G_n\Big(x+\frac{k}{m}, \frac{py}{m^2}, \lambda^m\Big)$$

Proof. We know

$$\sum_{n=0}^{\infty} G_n(mx,py,\lambda)\frac{t^n}{n!} = \frac{2te^{mxt+pyt^2}}{\lambda e^t+1}$$

and handing the R. H. S of the above equations, we defined

$$\sum_{n=0}^{\infty} G_n(mx,py,\lambda)\frac{t^n}{n!} = \frac{2te^{mxt}}{\lambda^m e^{mt}+1}\frac{\lambda^m e^{mt}+1}{\lambda e^t+1}e^{pyt^2}$$

By noting that

$$\frac{2te^{mxt}}{\lambda^m e^{mt}+1}\frac{\lambda^m e^{mt}+1}{\lambda e^t+1}e^{pyt^2}=\sum_{k=0}^{m-1}\frac{1}{m}(-1)^k\lambda^k\sum_{q=0}^{\infty}\frac{t^q m^q}{q!}G_q\Big(x+\frac{k}{m},\lambda^m\Big)\sum_{r=0}^{\infty}\frac{t^{2r}p^r}{r!}y^r$$

We get

$$\sum_{n=0}^{\infty}G_n(mx,py,\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}t^r m^{n-1}\sum_{k=0}^{m-1}(-1)^k\lambda^k\sum_{r=0}^{[\frac{n}{2}]}\frac{G_{n-2r}(x+\frac{k}{m},\lambda^m)}{(n-2r)!r!}\Big(\frac{py}{m^2}\Big)^r$$

Also, by simple computation we realize that

$${}_HG_n(x,y,\lambda)=\sum_{s=0}^{[\frac{n}{2}]}\frac{y^s G_{n-2s}(x,\lambda)}{s!(n-2s)!}$$

So, we obtain

$$G_n(mx,py,\lambda)=m^{n-1}\sum_{k=0}^{m-1}(-1)^k\lambda^k_H G_n\Big(x+\frac{k}{m},\frac{py}{m^2},\lambda^m\Big)$$

Therefore proof is complete. □

Also, by a similar method, we get the following remark.

Remark 4.2. *Let m be odd and $x,y\in\mathbb{R}^+$, we get*

$${}_HG_n(mx,m^2y,\lambda)=m^{n-1}\sum_{\ell=0}^{m-1}(-1)^\ell\lambda^\ell_H G_n\Big(x+\frac{\ell}{m},y,\lambda^m\Big).$$

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